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Super-Liouville — double Liouville correspondence

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Abstract: The AGT motivated relation between the tensor product of the $\mathcal{N} = 1$ super-Liouville field theory with the imaginary free fermion (SL) and a certain projected tensor product of the real and the imaginary Liouville field theories (LL) is analyzed. Using conformal field theory techniques we give a complete proof of the equivalence in the NS sector. It is shown that the SL-LL correspondence is based on the equivalence of chiral objects including suitably chosen chiral structure constants of all the three Liouville theories involved.

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Contents

1 Introduction

It was conjectured several years ago that partition functions of $\mathcal{N} = 2$ superconformal SU(N) gauge theories in four dimensions are directly related to correlation functions of the two-dimensional Liouville/Toda field theories [1, 2]. This by now well established AGT relation has been extended and generalized in various directions leading to many new developments on both sides of the correspondence. One of the essential generalizations was the proposal that $\mathcal{N} = 2$ SU(N) gauge theories on $\mathbb{R}^4/\mathbb{Z}_p$ should be related to certain coset conformal fields theories. This conjecture was first formulated in [3] where also the first checks in the case of $N = p = 2$ corresponding to the $\mathcal{N} = 1$ super-Liouville theory were presented. It was soon clarified that the general case $p > 2$ should correspond to para-Liouville/Toda theories [4]. Some further checks of the AGT relation for $N = p = 2$ where done in the NS sector in [5–7] and in the R sector in [8, 9]. It was in particular observed in [6, 7] that the blow-up formula for the Nekrasov partition function suggests a precise relation between $\mathcal{N} = 1$ super-Liouville and Liouville conformal blocks. An interesting explanation of this phenomenon on the CFT side was given in [10]. It was motivated by old results [11–14] relating various rational models realized as quotients,

$$V(p,m) \sim \frac{\widehat{\mathrm{SU}}(2)_p \times \widehat{\mathrm{SU}}(2)_m}{\widehat{\mathrm{SU}}(2)_{p+m}}.$$

The case relevant for the present paper is the relation between the Virasoro minimal models $V(m) = V(1,m)$ and the $\mathcal{N}=1$ superconformal models $SV(m) = V(2,m)$:

$$V(1) \otimes SV(m) \sim V(m) \otimes_P V(m+1), \qquad m = 1, 2, \ldots,$$

where the symbol $\otimes_P$ denotes a projected tensor product in which only selected pairs of conformal families are present. The nonrational counterpart of this relation proposed in [10] takes the schematic form

$$\text{free fermion} \otimes \mathcal{N}=1 \text{ super-Liouville} \quad \sim \quad \text{Liouville} \otimes_P \text{Liouville}. \tag{1.1}$$

In the NS sector this relation has been made much more precise in [15] where it was used as an essential element of the proof of the AGT correspondence in the case of $N = p = 2$. The considerations of [15] are based on the instanton partition function computed on the resolved space [6, 7]. It is interesting to note that a different instanton counting scheme for the ALE spaces, the orbifolded instanton counting, yields in general a slightly different partition function [16] and it is still an open question whether its CFT counterpart exists. The extension of (1.1) to the Ramond sector along with some nontrivial checks were presented in [17].

Although most of the ingredients and constructions were already discussed in [15] and [17] a precise formulation of (1.1) as an exact equivalence of CFT models is still an open problem. The product of the free fermion and the $\mathcal{N}=1$ super-Liouville theory (SL) is a perfectly well defined model of CFT but an exact definition of the double Liouville theory (LL) on the r.h.s. of (1.1) has not yet been clarified. There are also some essential elements of the proof missing. The aim of the present paper is to fill these gaps.

Let us briefly describe our proposal for the precise meaning of equivalence (1.1). The chiral symmetry algebra A_{NS} of the SL theory is the product of the $\mathcal{N}=1$ super-Virasoro algebra $\mathsf{SVir}_{\mathrm{NS}}$ and the Heisenberg algebra H_{NS} of fermion oscillators. One has therefore a $1\!-\!1$ correspondence between $\mathsf{SVir}_{\mathrm{NS}}$- and A_{NS}-primaries. Any highest weight representation $\mathcal{A}_{\Delta_p}$ of A_{NS} with the central charge $c = \frac{3}{2} + 3Q^2, Q = b + b^{-1}$ and the highest weight $\Delta_p = \frac{Q^2}{8} + \frac{p^2}{2}$ carries a representation of two mutually commuting Virsoro algebras $\{L_n^{\mathrm{L}}\}$, $\{L_n^{\Gamma}\}$ with the corresponding central charges given by [10–15]:

$$\begin{aligned} c^{\mathrm{L}} &= 1 + 6\,(Q^{\mathrm{L}})^2, &\qquad Q^{\mathrm{L}} &= b^{\mathrm{L}} + \frac{1}{b^{\mathrm{L}}}, &\qquad b^{\mathrm{L}} &= \frac{2b}{\sqrt{2-2b^2}}, \\ c^{\Gamma} &= 1 - 6\,(Q^{\Gamma})^2, &\qquad Q^{\Gamma} &= \frac{1}{b^{\Gamma}} - b^{\Gamma}, &\qquad \frac{1}{b^{\Gamma}} &= \frac{2}{\sqrt{2-2b^2}}. \end{aligned} \tag{1.2}$$

For real b they are on the opposite sides of the $c=1$ barrier. We assume $b<1$ throughout the paper. Then

$$c^{\mathrm{L}} > 25, \quad 1 > c^{\Gamma}$$

and the parameterizations above coincide with the standard ones of the Liouville theory [18–21] and of the generalized minimal models (GMM, the time-like Liouvile theory) [22, 23].

It was shown in [15] that the $\mathsf{Vir}\oplus\mathsf{Vir}$ representation on $\mathcal{A}_{\Delta_p}$ decomposes into irreducible components

$$\mathcal{A}_{\Delta_p} = \bigoplus_{j\in\mathbb{Z}} \mathsf{V}_{\Delta^{\mathrm{L}}(p,j)} \otimes \mathsf{V}_{\Delta^{\Gamma}(p,j)} \tag{1.3}$$

where $\mathsf{V}_{\Delta^{\mathrm{L}}(p,j)}$, $\mathsf{V}_{\Delta^{\Gamma}(p,j)}$ are Verma modules of the corresponding Virasoro algebras with the highest weights

$$\Delta^{\mathrm{L}}(p,j) = \frac{1}{1-b^2}\left(\frac{Q^2}{8} + \frac{(p+ij\,b)^2}{2}\right), \quad \Delta^{\Gamma}(p,j) = \frac{1}{1-b^{-2}}\left(\frac{Q^2}{8} + \frac{\left(p+ij\,b^{-1}\right)^2}{2}\right).$$

Decomposition (1.3) implies that the spectrum of the double Liouville theory, although diagonal in the continuous parameter p, is non-diagonal in the discrete index j. This important novelty requires appropriate off diagonal extensions of the DOZZ [18–20] and the GMM [22, 23] structure constants. A simple idea is to split the diagonal constants into chiral parts. As we shall see this receives strong support from our calculations. Indeed it turns out that the SL-LL equivalence is to large extend based on the relations between chiral structure constants.

In the standard normalization of the Liouville theory the reflection amplitude and the two-point function are equal [18–20]. This is suitable for the analytic continuation arguments and natural from the path integral point of view [24]. Other normalizations were discussed in the context of the GMM [22, 25]. In the present discussion it is convenient to choose the symmetric normalization

$$\Phi_\alpha = \Phi_{Q-\alpha}$$

with no restrictions on the two-point functions. This simplifies the relation between fields on both sides of the correspondence.

In the symmetric normalization the DOZZ structure constant for primary fields Φ_α can be written as

$$\begin{aligned}
C_b^{\mathrm{DOZZ}}(\alpha_3,\alpha_2,\alpha_1) &\equiv \Big\langle \Phi_{\alpha_3}(\infty,\infty)\Phi_{\alpha_2}(1,1)\Phi_{\alpha_1}(0,0)\Big\rangle_{\mathrm{L}} \\
&= M_b^{\mathrm{L}}\mathsf{C}_b^{\mathrm{L}}(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{C}}_b^{\mathrm{L}}(\alpha_3,\alpha_2,\alpha_1) \qquad (1.4)\\
\mathsf{C}_b^{\mathrm{L}}(\alpha_3,\alpha_2,\alpha_1) &= \frac{\Gamma_b\left(\alpha_{123}-Q\right)\prod\limits_{i<j}\Gamma_b\left(\alpha_{ij}\right)}{\prod\limits_i\sqrt{\Gamma_b\left(2\alpha_i\right)\Gamma_b\left(2Q-2\alpha_i\right)}}\,,\\
\bar{\mathsf{C}}_b^{\mathrm{L}}(\alpha_3,\alpha_2,\alpha_1) &= \frac{\Gamma_b\left(2Q-\alpha_{123}\right)\prod\limits_{i<j}\Gamma_b\left(Q-\alpha_{ij}\right)}{\prod\limits_i\sqrt{\Gamma_b\left(Q-2\alpha_i\right)\Gamma_b\left(2\alpha_i-Q\right)}}\,,
\end{aligned}$$

where $\alpha_{123}=\alpha_1+\alpha_2+\alpha_3$, $\alpha_{12}=\alpha_1+\alpha_2-\alpha_3$, etc.[1]

[1] A definition of the Barnes double gamma function along with some of its properties is given in appendix C.

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The same can be done for the GMM structure constant

$$\begin{aligned} C_b^{\text{GMM}}(\alpha_3,\alpha_2,\alpha_1) &\equiv \Big\langle \Phi_{\alpha_3}(\infty,\infty)\Phi_{\alpha_2}(1,1)\Phi_{\alpha_1}(0,0)\Big\rangle_{\Gamma} \\ &= M_b^{\Gamma}\mathsf{C}_b^{\Gamma}(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{C}}_b^{\Gamma}(\alpha_3,\alpha_2,\alpha_1), \end{aligned} \tag{1.5}$$

$$\mathsf{C}_b^{\Gamma}(\alpha_3,\alpha_2,\alpha_1) = \frac{\prod\limits_i \sqrt{\Gamma_b\left(b+2\alpha_i\right)\Gamma_b\left(2b^{-1}-b-2\alpha_i\right)}}{\Gamma_b\left(\alpha_{123}-b^{-1}+2b\right)\prod\limits_{i<j}\Gamma_b\left(\alpha_{ij}+b\right)},$$

$$\bar{\mathsf{C}}_b^{\Gamma}(\alpha_3,\alpha_2,\alpha_1) = \frac{\prod\limits_i \sqrt{\Gamma_b\left(b^{-1}-2\alpha_i\right)\Gamma_b\left(2b-b^{-1}+2\alpha_i\right)}}{\Gamma_b\left(2b^{-1}-b-\alpha_{123}\right)\prod\limits_{i<j}\Gamma_b\left(b^{-1}-\alpha_{ij}\right)}.$$

Decompositions (1.4) and (1.5) are based on the splitting of

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(b^{-1}+b-x)}$$

into the Barnes gamma factors and are to some extend arbitrary. What is important for further calculations is that one uses the same splitting of Υ_b for terms with the same arguments in the DOZZ and the GMM structure constants. Once splittings are chosen one can introduce the off-diagonal extensions of the Liouville structure constants:

$$C_b^{\sigma}(\alpha_3,\alpha_2,\alpha_1,\overline{\alpha}_3,\overline{\alpha}_2,\overline{\alpha}_1) = M_b^{\sigma}\mathsf{C}_b^{\sigma}(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{C}}_b^{\sigma}(\overline{\alpha}_3,\overline{\alpha}_2,\overline{\alpha}_1), \quad \sigma = \text{L},\Gamma. \tag{1.6}$$

We shall split in a similar manner the NS super-Liouville structure constants [26, 27] (see also [28, 29]). In the symmetric normalization the NS structure constants (which coincide with the SL structure constants) can be written as

$$\begin{aligned} C_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) &= \Big\langle \Phi_{\alpha_3}(\infty,\infty)\Phi_{\alpha_2}(1,1)\Phi_{\alpha_1}(0,0)\Big\rangle_{\text{SL}} \\ &= M_b^{\text{NS}}\mathsf{C}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{C}}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1), \end{aligned} \tag{1.7}$$

$$\begin{aligned} \widetilde{C}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) &= \Big\langle \Phi_{\alpha_3}(\infty,\infty)\mathcal{G}_{-\frac12}\overline{\mathcal{G}}_{-\frac12}\Phi_{\alpha_2}(1,1)\Phi_{\alpha_1}(0,0)\Big\rangle_{\text{SL}} \\ &= 2iM_b^{\text{NS}}\mathsf{D}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{D}}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1), \end{aligned} \tag{1.8}$$

where

$$\mathsf{C}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) = \frac{\Gamma_b^{\text{NS}}\left(\alpha_{123}-Q\right)\prod\limits_{i<j}\Gamma_b^{\text{NS}}\left(\alpha_{ij}\right)}{\prod\limits_i \sqrt{\Gamma_b^{\text{NS}}\left(2\alpha_i\right)\Gamma_b^{\text{NS}}\left(2Q-2\alpha_i\right)}},$$

$$\bar{\mathsf{C}}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) = \frac{\Gamma_b^{\text{NS}}\left(2Q-\alpha_{123}\right)\prod\limits_{i<j}\Gamma_b^{\text{NS}}\left(Q-\alpha_{ij}\right)}{\prod\limits_i \sqrt{\Gamma_b^{\text{NS}}\left(Q-2\alpha_i\right)\Gamma_b^{\text{NS}}\left(2\alpha_i-Q\right)}},$$

$$\mathsf{D}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) = \frac{\Gamma_b^{\text{R}}\left(\alpha_{123}-Q\right)\prod\limits_{i<j}\Gamma_b^{\text{R}}\left(\alpha_{ij}\right)}{\prod\limits_i \sqrt{\Gamma_b^{\text{NS}}\left(2\alpha_i\right)\Gamma_b^{\text{NS}}\left(2Q-2\alpha_i\right)}},$$

$$\bar{\mathsf{D}}_b^{\text{NS}}(\alpha_3,\alpha_2,\alpha_1) = \frac{\Gamma_b^{\text{R}}\left(2Q-\alpha_{123}\right)\prod\limits_{i<j}\Gamma_b^{\text{R}}\left(Q-\alpha_{ij}\right)}{\prod\limits_i \sqrt{\Gamma_b^{\text{NS}}\left(Q-2\alpha_i\right)\Gamma_b^{\text{NS}}\left(2\alpha_i-Q\right)}},$$

and

$$\Gamma_b^{\rm NS}(x) = \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right), \qquad \Gamma_b^{\rm R}(x) = \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right).$$

The modular properties of the toric partition function of the SL theory imply that the GSO projection on the even with respect to the common left and right parity subspace is a necessary consistency condition. This yields

$$\mathcal{H}^{\rm SL} = \int\limits_{\mathbb{R}_+} dp\, \left[\left(\mathcal{A}_{\Delta_p}\right)_{\rm even} \otimes \left(\overline{\mathcal{A}}_{\Delta_p}\right)_{\rm even} \oplus \left(\mathcal{A}_{\Delta_p}\right)_{\rm odd} \otimes \left(\overline{\mathcal{A}}_{\Delta_p}\right)_{\rm odd} \right]. \tag{1.9}$$

On the r.h.s. of (1.1) it corresponds to

$$\mathcal{H}^{\rm LL} = \int\limits_{\mathbb{R}_+} dp \bigoplus_{\substack{j,\bar{j}\,\in\,\mathbb{Z}\\ j+\bar{j}\,\in\,2\mathbb{Z}}} \left(\mathsf{V}_{\Delta^{\rm L}(p,j)} \otimes \overline{\mathsf{V}}_{\Delta^{\rm L}(p,\bar{j})}\right) \otimes \left(\mathsf{V}_{\Delta^{\Gamma}(p,j)} \otimes \overline{\mathsf{V}}_{\Delta^{\Gamma}(p,\bar{j})}\right). \tag{1.10}$$

The spaces of states $\mathcal{H}^{\rm SL}$, $\mathcal{H}^{\rm LL}$ and structure constants (1.6), (1.7), (1.8) unambiguously define both sides of the SL-LL correspondence in the NS sector. The main statement we are concerned with in this paper can be formulated as follows.

If the relative normalization condition

$$\frac{M_b^{\rm NS}}{M^{\rm L}_{b^{\rm L}} M^{\Gamma}_{b^{\Gamma}}} = \frac{\Upsilon_{b^{\Gamma}}(b^{\Gamma})\Upsilon_b(b)\,\Upsilon_b\left(\frac{Q}{2}\right)}{\Upsilon_{b^{\rm L}}(b^{\rm L})} b^{-\frac{b^2}{1-b^2}\frac{Q^2}{2}} \left(\frac{1-b^2}{2}\right)^{-\frac{Q^2}{4}+\frac{1}{2}} \tag{1.11}$$

is satisfied then there exists a map $\mathcal{I}$ from the algebra of local fields of the NS sector of the GSO projected SL model to the algebra of local fields of the GSO projected LL model preserving all correlation functions on the sphere:

$$\Big\langle \Phi_{\alpha_n}\ldots\Phi_{\alpha_2}\Phi_{\alpha_1}\Big\rangle_{\rm SL} = \Big\langle \mathcal{I}(\Phi_{\alpha_n})\ldots\mathcal{I}(\Phi_{\alpha_2})\mathcal{I}(\Phi_{\alpha_1})\Big\rangle_{\rm LL}.$$

Let us remark that our motivation goes beyond a self contained CFT proof of the statement above. It characterizes only the simplest of the relations suggested by the AGT correspondence [30] and by the analogy with the rational cases [10]. One may expect for instance an exact relation between the para-Liouville theories [31] and projected tensor product of several copies of the Liouville theory. It seems that better understanding of all equivalences of this type would provide a new insight into underlying structures of at least certain class of nonrational CFT models.

The organization of the paper is as follows. In section 2.1 the basic concepts of the free field realization of the NS sector are introduced mainly for the notational convenience. In section 2.2 a special attention is paid to the description of the reflection map and the reflected modes in the NS Verma modules which play an essential role in the constructions of [15]. The properties of the states generated by solely fermionic modes are summarized in Propositions 1 and 2 of subsection 2.2. They are proven in appendix A.

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Section 3 is devoted to the properties of basic chiral structures related to the chiral symmetry algebra $\mathsf{A}_{\rm NS}$ of the SL theory. In subsection 3.1 we review decomposition (1.3) of the $\mathsf{A}_{\rm NS}$ Verma module into projected tensor product of Virasoro Verma modules constructed in [15]. In subsection 3.2 we analyze 3-point $\mathsf{A}_{\rm NS}$ conformal blocks and derive the central for the whole paper formula for the blow up factor. This is the main and technically the most involved result of the present paper. It is based on Propositions 1 and 2 of subsection 2.2 and on the formula for generalized Selberg integral derived in appendix B. In subsection 3.3 we investigate the $\mathsf{A}_{\rm NS}$ algebra 4-point conformal blocks. As a side result we obtain expressions for the NS super-Virasoro blocks in terms of the Virasoro ones and the blow up factors.

In section 4 we complete our proof of the SL-LL equivalence for arbitrary correlation functions on the sphere in the GSO projected NS sector. To this end we analyze in subsection 4.1 relations between chiral structures of all the there theories involved. It turns out that one can prove the SL-LL equivalence for the left and for the right chiral parts separately. It is also remarkable that although the left and the right structure constants are different the relations they satisfy involve exactly the same coefficients. This is a consequence of two types of identities for Barnes gamma functions which are derived in appendix C. In section 4.2 we analyze how the relations between chiral structures result in an equivalence of full CFT theories. We construct an exact map between local fields and show it preserves all the 3-point functions and the factorization of correlation functions on the sphere.

There are several possible continuations of the present paper. The most obvious is a completion of the proof for the whole GSO projected models. With the results of [17] it seems that an extension to the Ramond sector is straightforward. The same concerns an extension to arbitrary closed surfaces which would require an analysis of 1-point toric functions. More challenging is an extension to bordered surfaces.

The chiral structure constants played an essential role in the calculations of the present paper.They also show up in the relative normalization of the fusing matrix and the 6j-symbol of a continuous series of representations of $\mathcal{U}_q(sl(2,\mathbb{R}))$ [21]. This relation and the orthogonality properties of the 6j-symbols yield a general expression for structure constants satisfying the crossing bootstrap equation on the Liouville theory spectrum ([21], formula (252)). To what extend the chiral structure constants form fundamental building blocks of Liouville type models is by itself important problem related to the questions of classification and new model building. The double Liouville theory is the first example of nonrational models with the spectrum being diagonal in the continuous and non-diagonal in the discrete parameter. It is interesting if there are other consistent models of this type.

Another possible topic is to investigate already mentioned relations between the para-Liouville theories and projected tensor products of several copies of the Liouville theory which are well supported by the AGT correspondence [6, 7, 10]. For rational models another equivalence is known [12]. It relates the tensor products of the $\mathcal{N}=2$ super minimal models and the Ising model to projected tensor products of the $\mathcal{N}=1$ super minimal models and the parafermionic models. Although the AGT 4-dimensional counterpart is not expected in this case it would be very interesting to check if there is a non-rational version of this correspondence.

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2 Free field representation of the NS sector

2.1 Chiral vertex operators

Let $\mathcal{F}_b$ be the bosonic Fock space generated by modes of the Heisenberg algebra

$$[c_m, c_n] = m\delta_{m+n,0}, \qquad m,n \in \mathbb{Z}\setminus\{0\}, \tag{2.1}$$

out of the vacuum $|0_b\rangle$ satisfying $c_m|0_b\rangle = 0$ for $m > 0$. We denote by $\mathcal{F}_{\rm NS}$ the fermionic NS Fock space generated by modes of the algebra

$$\{\psi_k, \psi_l\} = \delta_{k+l,0}, \qquad k,l \in \mathbb{Z}+\frac{1}{2}, \tag{2.2}$$

out of the vacuum $|0_f\rangle$ obeying $\psi_r|0_f\rangle = 0$ for $r > 0$. The super-scalar Hilbert space $\mathcal{H}_{\rm NS}$ of the NS sector can be introduced as a tensor product

$$\mathcal{H}_{\rm NS} = \mathcal{H}_0 \otimes \mathcal{F}_b \otimes \mathcal{F}_{\rm NS},$$

where $\mathcal{H}_0 \cong L^2(\mathbb{R})$ is a representation space of the zero mode operators p,q, satisfying $[\mathsf{p},\mathsf{q}] = -i$. A scalar product in $\mathcal{H}_{\rm NS}$ is defined by imposing modes' conjugation properties

$$\mathsf{p}^\dagger = \mathsf{p}, \qquad \mathsf{q}^\dagger = \mathsf{q}, \qquad c_n^\dagger = c_{-n}, \qquad \psi_k^\dagger = \psi_{-k}, \tag{2.3}$$

together with the normalization condition

$$\langle p_1|p_2\rangle = \delta(p_1 - p_2),$$

where $|p\rangle = |p\rangle\!\rangle \otimes |0_b\rangle \otimes |0_f\rangle$ and $\mathsf{p}|p\rangle\!\rangle = p|p\rangle\!\rangle$. The space $\mathcal{H}_{\rm NS}$ can be seen as a direct integral of Hilbert spaces:

$$\mathcal{H}_{\rm NS} = \int_{\mathbb{R}} dp\, \mathcal{H}_p\,, \qquad \mathcal{H}_p = |p\rangle\!\rangle \otimes \mathcal{F}_b \otimes \mathcal{F}_{\rm NS}, \tag{2.4}$$

with the scalar product in $\mathcal{H}_p$ determined by conjugation properties (2.3) and the normalization $\langle p|p\rangle = 1$. Vectors $c_{-M}\psi_{-K}|p\rangle$ where

$$c_{-M} = c_{-m_j}\ldots c_{-m_1}, \quad m_j \geqslant \ldots \geqslant m_1, \quad m_r \in \mathbb{N},$$
$$\psi_{-K} = \psi_{-k_i}\ldots\psi_{-k_1}, \quad k_i > \ldots > k_1, \quad k_s \in \mathbb{N} - \frac{1}{2},$$

form an orthogonal basis in $\mathcal{H}_p$,

$$\langle p|\,\psi^\dagger_{-K'}c^\dagger_{-M'}c_{-M}\psi_{-K}\,|p\rangle = N_{MK}\delta_{M,M'}\delta_{K,K'}.$$

The space $\mathcal{H}_{\rm NS}$ carries two representations $(\pm)$ of the NS algebra with the same central charge $c = \frac{3}{2} + 3Q^2$:

$$\begin{aligned}
L_0(\pm\mathsf{p}) &= \sum_{m\geqslant 1} c_{-m}c_m + \sum_{k\geqslant\frac{1}{2}} k\,\psi_{-k}\psi_k + \frac{1}{8}Q^2 + \frac{1}{2}\mathsf{p}^2,\\
L_n(\pm\mathsf{p}) &= \frac{1}{2}\sum_{m\neq 0,n} c_{n-m}c_m + \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} k\,\psi_{n-k}\psi_k + \left(\frac{inQ}{2} \pm \mathsf{p}\right)c_n, \qquad n\neq 0,\\
G_k(\pm\mathsf{p}) &= \sum_{m\neq 0} c_m\psi_{k-m} + (iQk \pm \mathsf{p})\psi_k,
\end{aligned} \tag{2.5}$$

and the same highest weight states

$$\begin{aligned}
L_0(\pm\mathsf{p})\,|p\rangle &= \Delta_p\,|p\rangle\,,\\
L_0(\pm\mathsf{p})\,|-p\rangle &= \Delta_p\,|-p\rangle\,, \qquad\qquad \Delta_p = \frac{1}{8}Q^2+\frac{1}{2}p^2,\\
L_n(\pm\mathsf{p})\,|p\rangle &= G_k(\pm\mathsf{p})\,|p\rangle = 0,\\
L_n(\pm\mathsf{p})\,|-p\rangle &= G_k(\pm\mathsf{p})\,|-p\rangle = 0, \quad n,k>0.
\end{aligned}$$

Let us define the superscalar components

$$\varphi_<(z) = -i\sum_{n=1}^{\infty}\frac{c_{-n}}{n}\,z^n, \qquad \varphi_>(z) = i\sum_{n=1}^{\infty}\frac{c_n}{n}\,z^{-n}, \qquad \psi(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\psi_k\,z^{-k-\frac{1}{2}}$$

and the ordered exponential

$$\mathsf{E}^{\alpha}(z) \;=\; z^{-\Delta_\alpha}\,\mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}}\,\mathrm{e}^{\alpha\varphi_<(z)}\,z^{-i\alpha\mathsf{p}}\,\mathrm{e}^{\alpha\varphi_>(z)}\,\mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}}, \tag{2.6}$$

where $\Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha)$. Explicit calculations give:

$$\begin{aligned}
[\mathsf{p},\mathsf{E}^{\alpha}(z)] &= -i\alpha\mathsf{E}^{\alpha}(z),\\
[L_n(\mathsf{p}),\mathsf{E}^{\alpha}(z)] &= z^n\big(z\partial_z+(n+1)\Delta_\alpha\big)\mathsf{E}^{\alpha}(z),\\
[G_k(\mathsf{p}),\mathsf{E}^{\alpha}(z)] &= -i\alpha\,z^{k+\frac{1}{2}}\,\psi(z)\mathsf{E}^{\alpha}(z),\\
[L_n(\mathsf{p}),\psi(z)\mathsf{E}^{\alpha}(z)] &= z^n\left(z\partial_z+(n+1)\left(\Delta_\alpha+\frac{1}{2}\right)\right)\psi(z)\mathsf{E}^{\alpha}(z),\\
\{G_k(\mathsf{p}),\psi(z)\mathsf{E}^{\alpha}(z)\} &= \frac{i}{\alpha}z^{k-\frac{1}{2}}\big(z\partial_z+(2k+1)\Delta_\alpha\big)\mathsf{E}^{\alpha}(z).
\end{aligned} \tag{2.7}$$

The ordered exponential is thus an NS super-primary field which can be seen as a family of maps

$$\mathsf{E}^{\alpha}(z):\;\; \mathcal{H}_p \;\mapsto\; \mathcal{H}_{p-i\alpha}.$$

In general

$$\mathsf{E}^{\alpha_n}(z_n)\ldots\mathsf{E}^{\alpha_1}(z_1):\;\; \mathcal{H}_p \;\mapsto\; \mathcal{H}_{p-i(\alpha_1+\ldots+\alpha_n)}.$$

Iff the *neutrality condition*

$$q = p - i(\alpha_1+\ldots\alpha_n) \tag{2.8}$$

holds the matrix elements of ordered exponentials

$$\langle q|\mathsf{E}^{\alpha_n}(z_n)\ldots\mathsf{E}^{\alpha_1}(z_1)|p\rangle$$

do not vanish and coincide with the chiral correlators of NS primary fields

$$\langle\Delta_q|\mathsf{V}_{\Delta_{\alpha_n}}(z_n)\ldots\mathsf{V}_{\Delta_{\alpha_1}}(z_1)|\Delta_p\rangle.$$

For $\alpha = b$ or $\alpha = \frac{1}{b}$:

$$\begin{aligned}
[L_n(\mathsf{p}),\psi(z)\mathsf{E}^{\alpha}(z)] &= z^n\big(z\partial_z+(n+1)\big)\psi(z)\mathsf{E}^{\alpha}(z) = \partial_z\big(z^{n+1}\psi(z)\mathsf{E}^{\alpha}(z)\big),\\
\{G_k(\mathsf{p}),\psi(z)\mathsf{E}^{\alpha}(z)\} &= \frac{i}{\alpha}z^{k-\frac{1}{2}}\left(z\partial_z+\left(k+\frac{1}{2}\right)\right)\mathsf{E}^{\alpha}(z) = \frac{i}{\alpha}\partial_z\left(z^{k+\frac{1}{2}}\mathsf{E}^{\alpha}(z)\right).
\end{aligned}$$

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Hence, for closed contours the screening charge operators

$$\mathsf{Q}_b = \oint dz\ \psi(z)\mathsf{E}^b(z) \qquad \text{and} \qquad \mathsf{Q}_{\frac{1}{b}} = \oint dz\ \psi(z)\mathsf{E}^{\frac{1}{b}}(z),$$

satisfy the relations

$$\left[L_m(\mathsf{p}), \mathsf{Q}_b\right] = \left\{G_k(\mathsf{p}), \mathsf{Q}_b\right\} = \left[L_m(\mathsf{p}), \mathsf{Q}_{\frac{1}{b}}\right] = \left\{G_k(\mathsf{p}), \mathsf{Q}_{\frac{1}{b}}\right\} = 0.$$

In the free field model one can represent matrix element of an arbitrary NS chiral primary field $V_{\Delta_\alpha}(z)$ in eight possible ways:

$$\begin{aligned}
\langle\Delta_q|V_{\Delta_\alpha}(z)|\Delta_p\rangle &= \langle\pm q|\mathsf{E}^{\alpha}(z)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|\pm p\rangle \qquad \text{for} \quad \pm q = \pm p - i\alpha - i\left(rb + sb^{-1}\right), \\
\langle\Delta_q|V_{\Delta_\alpha}(z)|\Delta_p\rangle &= \langle\pm q|\mathsf{E}^{Q-\alpha}(z)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|\pm p\rangle \qquad \text{for} \quad \pm q = \pm p + i\alpha - i\left((r+1)b + (s+1)b^{-1}\right),
\end{aligned} \tag{2.9}$$

with uncorrelated signs in front of p and q.

Under similar restrictions on the "momenta" p and q, there also exist free field representations of matrix elements of the operator

$$*V_{\Delta_\alpha}(1) = \mathcal{G}_{-\frac{1}{2}}V_{\Delta_\alpha}(1) = [G_{-\frac{1}{2}}, V_{\Delta_\alpha}(1)].$$

Using (2.7) we get

$$*V_{\Delta_\alpha}(1) = -i\alpha\psi(1)\mathsf{E}^{\alpha}(1)$$

hence

$$\langle\Delta_q|*V_{\Delta_\alpha}(z)|\Delta_p\rangle = -i\alpha\langle\pm q|\psi(1)\mathsf{E}^{\alpha}(z)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|\pm p\rangle \tag{2.10}$$

for $\pm q = \pm p - i\alpha - i\left(rb + sb^{-1}\right)$ and

$$\langle\Delta_q|*V_{\Delta_\alpha}(z)|\Delta_p\rangle = -i(Q-\alpha)\langle\pm q|\psi(1)\mathsf{E}^{Q-\alpha}(z)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|\pm p\rangle \tag{2.11}$$

for $\pm q = \pm p + i\alpha - i\left((r+1)b + (s+1)b^{-1}\right)$. Since the fermion field $\psi(z)$ and the screening charges are odd with respect to the fermion parity operator $(-1)^F$ defined by

$$\left\{(-1)^F, \psi_k\right\} = \left[(-1)^F, c_m\right] = 0,$$

the total number of screening charges appearing in (2.9) must be even, while the total number of screening charges appearing in (2.10) and in (2.11) must be odd.

2.2 Reflection map in the NS sector

For each level $t \in \frac{1}{2}\mathbb{N}$ we introduce the transition matrices $S^t(\pm p)$:

$$\begin{aligned}
L_{-M}(\pm\mathsf{p})G_{-K}(\pm\mathsf{p})\,|p\rangle &= \sum_{|N|+|L|=t} S^t_{NL,MK}(\pm p)\, c_{-N}\psi_{-L}\,|p\rangle\,, \\
S^t_{NL,MK}(\pm p) &= \frac{\langle p|\,\psi^\dagger_{-L}c^\dagger_{-N}L_{-M}(\mathsf{p})G_{-K}(\mathsf{p})\,|\pm p\rangle}{N_{NL}}.
\end{aligned} \tag{2.12}$$

The matrix S^t (with a different normalization) was studied in [32] where the formula for its determinant was found. It takes the form

$$\det S^t(\pm p) = \text{const} \prod_{\substack{1\leqslant rs\leqslant 2t\\ r+s\in 2\mathbb{N}}} (p \mp p_{rs})^{P_{\rm NS}(n-\frac{rs}{2})}\,, \qquad p_{rs} = i\Big(rb + \frac{s}{b}\Big)\,, \qquad Q = b + \frac{1}{b}, \tag{2.13}$$

where multiplicities $P_{\rm NS}(f)$ are defined by the generating function

$$\sum_{f\in\frac{1}{2}\mathbb{N}} P_{\rm NS}(f)\, x^f = \prod_{m\in\mathbb{N}} \frac{1}{1-x^m} \prod_{k\in\mathbb{N}-\frac{1}{2}} (1+x^k)\,.$$

For real Q and p

$$L_n(p)^\dagger = L_{-n}(p), \qquad G_k(p)^\dagger = G_{-k}(p), \tag{2.14}$$

and $S^t(\pm p)$ are simply related to the Gram matrix B^t of the t-level Schapovalov form

$$\begin{aligned} B^t(\Delta_p) &= S^t(\pm p)^\dagger N^t S^t(\pm p), \\ B^t(\Delta_p)_{M'K',MK} &= \langle \Delta_p |\, G^\dagger_{-K'} L^\dagger_{-M'} L_{-M} G_{-K}\, |\Delta_p\rangle\,. \end{aligned} \tag{2.15}$$

Let $|\Delta_p\rangle$ be the normalized highest weight vector in the NS Verma module $\mathcal{V}_{\Delta_p}$ of the highest weight Δ_p. The maps

$$\begin{aligned} \imath_\pm(p) :\ \mathcal{V}_{\Delta_p} \ni L_{-M}G_{-K}\,|\Delta_p\rangle &\to L_{-M}(\mathsf{p})G_{-K}(\mathsf{p})\,|\pm p\rangle \in \mathcal{H}_{\pm p}, \\ \jmath_\pm(p) :\ \mathcal{V}_{\Delta_p} \ni L_{-M}G_{-K}\,|\Delta_p\rangle &\to L_{-M}(-\mathsf{p})G_{-K}(-\mathsf{p})\,|\pm p\rangle \in \mathcal{H}_{\pm p} \end{aligned} \tag{2.16}$$

are by construction morphisms of representations. Determinant formula (2.13) and hermitian conjugation properties (2.14) imply that for real Q and p these maps are unitary isomorphisms. So is the composition

$$r(p) = \imath_-(p) \circ \imath_+(p)^{-1} : \mathcal{H}_p \to \mathcal{H}_{-p}$$

called the reflection map [19, 21].[2] Up to a multiplicative constant it is uniquely determined by the intertwining property

$$L_m(-p)r(p) = r(p)L_m(p), \quad G_k(-p)r(p) = r(p)G_k(p). \tag{2.17}$$

One can study the relation between representations $\{L(+p), G(+p)\}$ and $\{L(-p), G(-p)\}$ introducing reflected modes in $\mathcal{H}_p$ [15] (see also [33] for a similar construction in the Virasoro case):

$$c^{\rm R}_m(p) = r(p)^{-1} c_m\, r(p), \qquad \psi^{\rm R}_k(p) = r(p)^{-1}\psi_k\, r(p). \tag{2.18}$$

[2]In the present construction $r(p)\,|p\rangle = |-p\rangle$ which is in line with the symmetric normalization of the structure constants. In the usual formulation of Liouville theory it is more convenient to normalize the reflection map by the condition $r(p)\,|p\rangle = D_{\rm NS}\left(\frac{Q}{2}+ip\right)|-p\rangle$ where $D_{\rm NS}$ denotes the NS reflection amplitude [26, 27].

In principle they can be expressed as series of monomials in c_m and ψ_r. The general form of this transformation is however not known. It can be calculated term by term from intertwining property (2.17) which takes the form:

$$\begin{aligned}
&\tfrac{1}{2}\sum_{m\neq 0,n} c_{n-m}c_m + \tfrac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} k\,\psi_{n-k}\psi_k + \left(\tfrac{inQ}{2}+p\right)c_n \\
&\qquad\qquad = \tfrac{1}{2}\sum_{m\neq 0,n} c^{\mathrm{R}}_{n-m}c^{\mathrm{R}}_m + \tfrac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} k\,\psi^{\mathrm{R}}_{n-k}\psi^{\mathrm{R}}_k + \left(\tfrac{inQ}{2}-p\right)c^{\mathrm{R}}_n \\
&\sum_{k\neq 0} c_k\psi_{l-k} + \big(iQl+p\big)\psi_l = \sum_{k\neq 0} c^{\mathrm{R}}_k\psi^{\mathrm{R}}_{l-k} + \big(iQl-p\big)\psi^{\mathrm{R}}_l.
\end{aligned} \tag{2.19}$$

The matrix of the reflection map $r^t(p)$ with respect to the basis $c_{-M}\psi_{-K}\,|p\rangle$

$$c^{\mathrm{R}}_{-N}(p)\psi^{\mathrm{R}}_{-L}(p)\,|p\rangle = \sum_{N'L'} r^t(p)_{N'L',NL}\,c_{-N'}\psi_{-L'}\,|p\rangle$$

is simply related to the transition matrix

$$r^t(p) = S^t(-p)^{-1}S^t(p).$$

We shall also need expressions of the oscillation bases $\{c_{-N}\psi_{-L}\,|p\rangle\}$, $\{c^{\mathrm{R}}_{-N}(p)\psi^{\mathrm{R}}_{-L}(p)\,|p\rangle\}$ in terms of $\{L_{-M}(p)G_{-K}(p)\,|p\rangle\}$:

$$\begin{aligned}
c_{-N}\psi_{-L}\,|p\rangle &= \sum_{MK} S^t(p)^{-1}_{MK,NL}L_{-M}(p)G_{-K}(p)\,|p\rangle \\
&= \sum_{MK,M'K'} N^t_{NL}\overline{S^t(p)}_{NL,M'K'}\,B^t(\Delta_p)^{-1}_{M'K',MK}L_M(p)G_{-K}(p)\,|p\rangle\,, \\
c^{\mathrm{R}}_{-N}(p)\psi^{\mathrm{R}}_{-L}(p)\,|p\rangle &= \sum_{MK} S^t(-p)^{-1}_{MK,NL}L_{-M}(p)G_{-K}(p)\,|p\rangle \\
&= \sum_{MK,M'K'} N^t_{NL}\overline{S^t(-p)}_{NL,M'K'}\,B^t(\Delta_p)^{-1}_{M'K',MK}L_M(p)G_{-K}(p)\,|p\rangle\,.
\end{aligned} \tag{2.20}$$

In the Fock space representation of an NS Verma module we used above the modes c_m, ψ_k are elementary p-independent objects in contrast to their reflected counterparts $c^{\mathrm{R}}_m(p), \psi^{\mathrm{R}}_k(p)$ which are complicated infinite series with coefficients being rational functions of momentum p. A symmetric picture can be achieved by lifting both sets of modes to the NS Verma module:

$$\begin{aligned}
\tilde{c}_m(p) &= \imath_+(p)^{-1}c_m\,\imath_+(p), & \tilde{\psi}_k(p) &= \imath_+(p)^{-1}\psi_k\,\imath_+(p), \\
\tilde{c}^{\mathrm{R}}_m(p) &= \imath_-(p)^{-1}c_m\,\imath_-(p), & \tilde{\psi}^{\mathrm{R}}_k(p) &= \imath_-(p)^{-1}\psi_k\,\imath_-(p).
\end{aligned} \tag{2.21}$$

In this formulation

$$\tilde{c}^{\mathrm{R}}_m(p) = \tilde{c}_m(-p), \quad \tilde{\psi}^{\mathrm{R}}_k(p) = \tilde{\psi}_k(-p), \tag{2.22}$$

and the reflection map takes the form:

$$\tilde{r}(p) = \imath_+(p)^{-1}\imath_-(p)\;:\;\mathcal{V}_{\Delta_p}\;\to\;\mathcal{V}_{\Delta_p}.$$

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Modes (2.21) can be used to construct two different orthogonal bases in the NS Verma module $\mathcal{V}_{\Delta_p}$

$$
\begin{aligned}
\tilde{c}_{-N}(p)\tilde{\psi}_{-L}(p)\left|\Delta_p\right\rangle &= \sum_{MK} S^t(p)^{-1}_{MK,NL} L_{-M}G_{-K}\left|\Delta_p\right\rangle \\
&= \sum_{MK,M'K'} N^n_{NL}\overline{S^t(p)}_{NL,M'K'}\, B^t(p)^{-1}_{M'K',MK} L_{-M}G_{-K}\left|\Delta_p\right\rangle, \\
\tilde{c}^{\mathrm{R}}_{-N}(p)\tilde{\psi}^{\mathrm{R}}_{-L}(p)\left|\Delta_p\right\rangle &= \sum_{MK} S^t(-p)^{-1}_{MK,NL} L_{-M}G_{-K}\left|\Delta_p\right\rangle \\
&= \sum_{MK,M'K'} N^t_{NL}\overline{S^t(-p)}_{NL,M'K'}\, B^t(p)^{-1}_{M'K',MK} L_{-M}G_{-K}\left|\Delta_p\right\rangle .
\end{aligned}
\tag{2.23}
$$

In a generic case (when the modules $\mathcal{V}_{\Delta_{rs}+\frac{rs}{2}}$ are irreducible) the coefficients in (2.23) have simple poles at $p = p_{rs}$ and $p = -p_{rs}$ respectively. In the following we shall need a more detailed information about the structure of pure fermionic states $\tilde{\psi}(p)_{-K}\left|\Delta_p\right\rangle$. This can be conveniently summarized in the form of two propositions below. The proofs are given in appendix A.

Proposition 1. *The only possible singularities of the coefficients of the decomposition of the state $\tilde{\psi}(p)_{-k_m}\ldots\tilde{\psi}_{-k_1}(p)\left|\Delta_p\right\rangle$, $k_m > \ldots > k_1$, with respect to the base $L_{-N}G_{-L}\left|\Delta_p\right\rangle$ are simple poles at*

$$
p = p_{rs}, \quad r+s \leqslant 2k_m+1, \quad r,s\in\mathbb{N}, \quad r+s\in 2\mathbb{N}.
$$

Let us define polynomials

$$
\begin{aligned}
\Omega(p,j) &= \begin{cases} l^{\mathrm{NS}}(2ip+Q,2j) & j>0 \\ l^{\mathrm{NS}}(-2ip+Q,-2j) & j<0 \end{cases} \\
l^{\mathrm{NS}}(x,n) &= \prod_{\substack{0\leqslant r,s \\ r+s<n \\ r+s\,\in\,2\mathbb{N}}} \left(x+rb+s\frac{1}{b}\right).
\end{aligned}
\tag{2.24}
$$

Proposition 1 implies that all coefficients of the decompositions

$$
\Omega(p,2k_m+1)\,\tilde{\psi}(p)_{-K}\left|\Delta_p\right\rangle = \sum a_{NL}(p)\,L_{-N}G_{-L}\left|\Delta_p\right\rangle \tag{2.25}
$$

are polynomials in p variable. One can estimate the degree of these polynomials.

Proposition 2. *Let $J = \{-\frac{2j-1}{2},\ldots,-\frac{1}{2}\}$ and let $K\subset J$. Then*

$$
\deg\left(\Omega(p,j)S^t(p)^{-1}_{NL,\emptyset K}\right) \leqslant \deg\Omega(p,j) - 2\#N - \#L
$$

where $t = |K|$.

3 Conformal blocks

3.1 Verma modules

Let $\mathcal{V}_{\Delta_p}$ be the NS module with the central charge $c = \frac{3}{2} + 3Q^2, Q = b + b^{-1}$ and the highest weight $\Delta_p = \frac{Q^2}{8} + \frac{p^2}{2}$. We introduce another copy of the fermionic NS Fock space $\widetilde{\mathcal{F}}_{\rm NS}$ generated by the modes

$$\{f_k, f_l\} = \delta_{k+l,0}, \quad k, l \in \mathbb{Z} + \frac{1}{2}$$

out of the vacuum state $|\,0_{\tilde{f}}\,\rangle$, $f_k|\,0_{\tilde{f}}\,\rangle = 0,\ k > 0$. The generators

$$L_n^{\mathsf{f}} = \frac{1}{2}\sum_k k :f_{n-k}f_k: \tag{3.1}$$

define on $\widetilde{\mathcal{F}}_{\rm NS}$ a representation of the $c = \frac{1}{2}$ Virasoro algebra which is a direct sum of two irreducible highest weight representations

$$\widetilde{\mathcal{F}}_{\rm NS} = \mathsf{V}_0 \oplus \mathsf{V}_{\frac{1}{2}} \tag{3.2}$$

with the highest weights $\Delta = 0$ and $\Delta = \frac{1}{2}$, respectively.

We shall introduce an indefinite scalar product on $\widetilde{\mathcal{F}}_{\rm NS}$ by the relations

$$f_k^\dagger = -f_{-k}, \quad \langle\,0_{\tilde{f}}\,|\,0_{\tilde{f}}\,\rangle = 1.$$

With respect to this scalar product $\left(L_n^{\mathsf{f}}\right)^\dagger = L_{-n}^{\mathsf{f}}$ and direct sum (3.2) is orthogonal with the induced scalar product positively definite on the first summand and negatively definite on the second.

On the $\mathsf{A}_{\rm NS}$ algebra Verma module $\mathcal{A}_{\Delta_p} = \mathcal{V}_{\Delta_p} \otimes \widetilde{\mathcal{F}}_{\rm NS}$ one can construct two sets of generators:

$$\begin{aligned} L_n^{\rm L} &= \frac{1}{1-b^2}L_n - \frac{1+2b^2}{1-b^2}L_n^{\mathsf{f}} + \frac{b}{1-b^2}\sum_r f_{n-r}G_r, \\ L_n^{\Gamma} &= \frac{1}{1-b^{-2}}L_n - \frac{1+2b^{-2}}{1-b^{-2}}L_n^{\mathsf{f}} + \frac{b^{-1}}{1-b^{-2}}\sum_r f_{n-r}G_r. \end{aligned} \tag{3.3}$$

They form two mutually commuting Virasoro algebras [11–13]

$$\begin{aligned} [L_m^{\rm L}, L_n^{\rm L}] &= (m-n)L_{m+n}^{\rm L} + \frac{c^{\rm L}}{12}(m^3-m)\delta_{m+n,0}, \\ [L_m^{\Gamma}, L_n^{\Gamma}] &= (m-n)L_{m+n}^{\Gamma} + \frac{c^{\Gamma}}{12}(m^3-m)\delta_{m+n,0}, \\ [L_m^{\rm L}, L_n^{\Gamma}] &= 0 \end{aligned}$$

and satisfy the standard conjugation relations:

$$(L_n^{\rm L})^\dagger = L_{-n}^{\rm L}, \quad (L_n^{\Gamma})^\dagger = L_{-n}^{\Gamma}. \tag{3.4}$$

The corresponding central charges are given by (1.2).

The problem of decomposing the $\mathsf{Vir} \oplus \mathsf{Vir}$ representation on $\mathcal{A}_{\Delta_p}$ into irreducible components has been analyzed in [15]. We shall briefly recall the main points of this derivation. Using construction (2.23) in the NS Verma module $\mathcal{V}_{\Delta_p}$ one introduces a family of states:

$$\begin{aligned}
|p,0\rangle &= |\Delta_p\rangle \otimes |\,0_{\tilde f}\,\rangle \in \mathcal{A}_{\Delta_p}, \\
|p,j\rangle &= \tilde\chi_{-\frac{2j-1}{2}}(p) \ldots \tilde\chi_{-\frac{3}{2}}(p)\,\tilde\chi_{-\frac{1}{2}}(p)\,|p,0\rangle \quad \text{for } j>0, \\
|p,j\rangle &= \tilde\chi^{\rm R}_{-\frac{2|j|-1}{2}}(p) \ldots \tilde\chi^{\rm R}_{-\frac{3}{2}}(p)\,\tilde\chi^{\rm R}_{-\frac{1}{2}}(p)\,|p,0\rangle \quad \text{for } j<0,
\end{aligned} \tag{3.5}$$

where $j \in \mathbb{Z}$ and

$$\tilde\chi_k(p) = f_k - i\tilde\psi_k(p), \qquad \tilde\chi^{\rm R}_k(p) = f_k - i\tilde\psi^{\rm R}_k(p), \qquad k \in \mathbb{Z} + \frac{1}{2}.$$

The modes can be conveniently organized into local fields

$$\begin{aligned}
\tilde\psi(\xi)|_{\mathcal{V}_{\Delta_p}\otimes\tilde{\mathcal{F}}_{\rm NS}} &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \tilde\psi_k(p)\xi^{-k-\frac{1}{2}}, \;\; \tilde\psi^{\rm R}(\xi)|_{\mathcal{V}_{\Delta_p}\otimes\tilde{\mathcal{F}}_{\rm NS}} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \tilde\psi^{\rm R}_k(p)\xi^{-k-\frac{1}{2}}, \\
\tilde\chi(\xi)|_{\mathcal{V}_{\Delta_p}\otimes\tilde{\mathcal{F}}_{\rm NS}} &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \tilde\chi_k(p)\xi^{-k-\frac{1}{2}}, \;\; \tilde\chi^{\rm R}(\xi)|_{\mathcal{V}_{\Delta_p}\otimes\tilde{\mathcal{F}}_{\rm NS}} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \tilde\chi^{\rm R}_k(p)\xi^{-k-\frac{1}{2}}.
\end{aligned} \tag{3.6}$$

Note that (2.22) implies $\tilde\chi^{\rm R}_k(p) = \tilde\chi_k(-p)$ hence

$$|p,-j\,\rangle = |{-p},j\,\rangle\,.$$

By construction

$$\tilde\chi_k(p)^\dagger = -\tilde\chi_k(p), \quad \tilde\chi^{\rm R}_k(p)^\dagger = -\tilde\chi^{\rm R}_k(p)$$

and

$$\{\tilde\chi_k(p), \tilde\chi_l(p)\} = \{\tilde\chi^{\rm R}_k(p), \tilde\chi^{\rm R}_l(p)\} = 0 \quad \text{for all } \; k,l \in \mathbb{Z}+\frac{1}{2}.$$

It follows that for all $j \in \mathbb{Z}$ the states $|p,j\rangle$ are of zero norm $(\langle\, p,j\;|p,j\,\rangle = 0)$.

In the Fock space $\mathcal{F}_{\rm NS} \otimes \tilde{\mathcal{F}}_{\rm NS}$ the modes $\tilde\chi_k(p), \tilde\chi^{\rm R}_k(p)$ are represented by

$$\chi_k = f_k - i\psi_k, \qquad \chi^{\rm R}_k(p) = f_k - i\psi^{\rm R}_k(p), \qquad k \in \mathbb{Z} + \frac{1}{2}.$$

With the help of both representations (2.19) one can compute the commutation relations

$$\begin{aligned}
[L^{\rm L}_n + L^{\Gamma}_n, \chi_r] &= -\left(\frac{1}{2}n + r\right)\chi_{r+n}, \\
[bL^{\rm L}_n + b^{-1}L^{\Gamma}_n, \chi_r] &= -\left((n+r)Q + ip\right)\chi_{r+n} + i\sum_{m\neq 0} c_m \chi_{r+n-m}, \\
[L^{\rm L}_n + L^{\Gamma}_n, \chi^{\rm R}_r] &= -\left(\frac{1}{2}n + r\right)\chi^{\rm R}_{r+n}, \\
[bL^{\rm L}_n + b^{-1}L^{\Gamma}_n, \chi^{\rm R}_r] &= -\left((n+r)Q - ip\right)\chi^{\rm R}_{r+n} + i\sum_{m\neq 0} c_m \chi^{\rm R}_{r+n-m},
\end{aligned}$$

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where

$$L_n^{\mathrm{L}} + L_n^{\Gamma} = L_n + \frac{1}{2}\sum_{k=-\infty}^{\infty} k : f_{n-k} f_k :,$$

$$b\, L_n^{\mathrm{L}} + b^{-1}\, L_n^{\Gamma} = Q \sum_{k=-\infty}^{\infty} k : f_{n-k} f_k : - \sum_{k=-\infty}^{\infty} f_{n-k} G_k.$$

Using these formulae one shows that states (3.5) are highest weight states with respect to both Virasoro algebras

$$\begin{aligned} L_0^{\mathrm{L}}\,|p,j\,\rangle &= \Delta^{\mathrm{L}}(p,j)\,|p,j\,\rangle\,, \quad L_n^{\mathrm{L}}\,|p,j\,\rangle = 0 \quad \text{for} \quad n>0, \\ L_0^{\Gamma}\,|p,j\,\rangle &= \Delta^{\Gamma}(p,j)\,|p,j\,\rangle\,, \quad L_n^{\Gamma}\,|p,j\,\rangle = 0 \quad \text{for} \quad n>0, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \Delta^{\mathrm{L}}(p,j) &= \alpha^{\mathrm{L}}(Q^{\mathrm{L}} - \alpha^{\mathrm{L}}) = \tfrac{1}{4}\,(Q^{\mathrm{L}})^2 + \left(p^{\mathrm{L}} + \tfrac{i}{2} j\, b^{\mathrm{L}}\right)^2 = \tfrac{1}{1-b^2}\left(\tfrac{Q^2}{8} + \tfrac{(p+ij\,b)^2}{2}\right), \\ \Delta^{\Gamma}(p,j) &= \alpha^{\Gamma}(\alpha^{\Gamma} - Q^{\Gamma}) = -\tfrac{1}{4}\,(Q^{\Gamma})^2 - \left(p^{\Gamma} + \tfrac{i}{2}\,\tfrac{j}{b^{\Gamma}}\right)^2 = \tfrac{1}{1-b^{-2}}\left(\tfrac{Q^2}{8} + \tfrac{\left(p+ij\,b^{-1}\right)^2}{2}\right), \\ \alpha^{\mathrm{L}} &= \tfrac{\alpha}{\sqrt{2-2b^2}} = \tfrac{Q^{\mathrm{L}}}{2} + ip^{\mathrm{L}}, \qquad p^{\mathrm{L}} = \tfrac{p}{\sqrt{2-2b^2}}, \\ \alpha^{\Gamma} &= \tfrac{b\,\alpha}{\sqrt{2-2b^2}} = \tfrac{Q^{\mathrm{L}}}{2} + ip^{\Gamma}, \qquad p^{\Gamma} = \tfrac{b\,p}{\sqrt{2-2b^2}}. \end{aligned} \tag{3.8}$$

By explicit calculation of vectors $\psi_{-\frac{1}{2}}(p)\,|p\rangle\,, \psi^{\mathrm{R}}_{-\frac{1}{2}}(p)\,|p\rangle$ one can show that $|p,j\,\rangle$ and $|p,-j\,\rangle$ are linearly independent. Since

$$\Delta^{\mathrm{L}}(p,j) + \Delta^{\Gamma}(p,j) = \frac{Q^2}{8} + \frac{p^2}{2} + \frac{j^2}{2} = \Delta_p + \frac{j^2}{2}\,, \tag{3.9}$$

one has two different Virasoro highest weight vectors on each $\frac{j^2}{2}$ level subspace of $\mathcal{A}_{\Delta_p}$ except $j=0$ where there is only one state $|p,0\rangle$. We have shown that

$$\bigoplus_{j\in\mathbb{Z}} \mathsf{V}_{\Delta^{\mathrm{L}}(p,j)} \otimes \mathsf{V}_{\Delta^{\Gamma}(p,j)} \;\subset\; \mathcal{A}_{\Delta_p}.$$

In order to show that the spaces above are equal one can calculate characters of the representations involved. For real p and Q the modules are non-degenerate and

$$\begin{aligned} \chi\left(\mathcal{V}_{\Delta_p}\right) &= q^{\Delta_p} \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n}, \\ \chi\left(\widetilde{\mathcal{F}}_{\mathrm{NS}}\right) &= \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}), \\ \chi\left(\mathsf{V}_{\Delta^{\sigma}(p,k)}\right) &= q^{\Delta^{\sigma}(p,k)} \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \quad \sigma = \mathrm{L}, \Gamma. \end{aligned}$$

Using relation (3.9) and the Jacobi triple identity one gets

$$\sum_{k=-\infty}^{+\infty} \chi\left(\mathsf{V}_{\Delta^{\mathrm{L}}(p,k)}\right) \chi\left(\mathsf{V}_{\Delta^{\Gamma}(p,k)}\right) = \chi\left(\mathcal{V}_{\Delta_p}\right) \chi\left(\widetilde{\mathcal{F}}_{\mathrm{NS}}\right),$$

hence

$$\bigoplus_{j\in\mathbb{Z}} \mathsf{V}_{\Delta^{\mathrm{L}}(p,j)} \otimes \mathsf{V}_{\Delta^{\Gamma}(p,j)} = \mathcal{A}_{\Delta_p}$$

which is just decomposition (1.3). The equivalence above is a unitary isomorphism if we assume on the l.h.s. the scalar product such that

$$\langle \nu^{\mathrm{L}}_{p,j} \otimes \nu^{\Gamma}_{p,j} \,|\, \nu^{\mathrm{L}}_{p,j'} \otimes \nu^{\Gamma}_{p,j'} \rangle = \langle\, p,j \;| p,-j\,\rangle\, \delta_{j+j',0} \tag{3.10}$$

where $\nu^{\sigma}_{p,j}$ denotes the highest weight state in the Virasoro Verma module $\mathsf{V}_{\Delta^{\sigma}(p,j)}$ ($\sigma = \mathrm{L}, \Gamma$). Let us stress that the skew form of product (3.10) is the only one consistent with the complex weights $\Delta^{\mathrm{L}}(p,k), \Delta^{\Gamma}(p,k), k \neq 0$ and the hermiticity of $L^{\mathrm{L}}_0, L^{\Gamma}_0$. We shall discuss this point in some more details in subsection 3.3.

3.2 Blow up factor

The three-point block $\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_3,\xi_2,\xi_1|z)$ with respect to the A_{NS} symmetry algebra is defined as a solution to the A_{NS} Ward identities normalized by the conditions:

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\nu_{p_3},\nu_{p_2},\nu_{p_1}|z) = \rho^{\mathrm{A}}_{\mathrm{NS}}(\nu_{p_3},*\nu_{p_2},\nu_{p_1}|z) = 1, \tag{3.11}$$

where $\nu_p = |\Delta_p\rangle \otimes |\,0_{\tilde f}\,\rangle$ and $*\nu_p = G_{-\frac12}\nu_p$.

Our goal in this subsection is to calculate the block $\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_3,\xi_2,\xi_1|z)$ for $z=1$ and for arbitrary states

$$\xi_{p,j} \equiv |p,j\,\rangle_n = \Omega(p,j)\,|p,j\,\rangle\,, \quad j \in \mathbb{Z}, \quad \Omega(p,0) = 1.$$

By definition

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = \begin{cases} \dfrac{{}_n\langle p_3,j_3|\,V_{p_2,j_2}(1)\,|p_1,j_1\,\rangle_n}{{}_n\langle p_3,0|\,V_{p_2,0}(1)\,|p_1,0\,\rangle_n} & \text{for } j_1+j_2+j_3 \in 2\mathbb{Z}, \\[2ex] \dfrac{{}_n\langle p_3,j_3|\,V_{p_2,j_2}(1)\,|p_1,j_1\,\rangle_n}{{}_n\langle p_3,0|\,*V_{p_2,0}(1)\,|p_1,0\,\rangle_n} & \text{for } j_1+j_2+j_3 \in 2\mathbb{Z}+1, \end{cases} \tag{3.12}$$

where $V_{p,j}(z)$ are chiral vertex operators corresponding to the states $|p,j\,\rangle_n$ and

$$*V_{p,0}(1) = \mathcal{G}_{-\frac12}V_{p,0} = [G_{-\frac12}, V_{p,0}(1)].$$

For $j=0$, $V_{p,0}(z) = V_{\Delta_p}\otimes 1$ where $V_{\Delta_p}(z)$ is the super-Virasoro primary field corresponding to the highest weight state $|\Delta_p\rangle$. An explicit expression for (3.12) was proposed in [15] where it was called the blow up factor due to the role it plays on the four-dimensional side of the AGT correspondence. We shall compute it employing the free field techniques. At the first step we show that (3.12) is a polynomial and calculate an upper bound on its degree. This is based on the propositions of subsection 2.2 proven in appendix A. Then using possible free filed representations we find all zeros of this polynomial. Finally in order to fix the overall constant we calculate (3.12) in a simple special case. The last step requires formulae for generalized Selberg integrals which are derived in appendix B.

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From decompositions (2.25) it follows that (3.12) is a polynomial in the parameters α_i, $i = 1, 2, 3$ with all α_i dependence coming from the Ward identities in the super-Liouville factor. To determine its order observe that for arbitrary multi-indices:

$$\deg_{\alpha_i}\left(\frac{\langle\,\Delta_{p_3}\,|\,G^\dagger_{-K_3}L^\dagger_{-M_3}\mathcal{L}_{-M_2}\mathcal{G}_{-K_2}V_{\Delta_{p_2}}(1)L_{-M_1}G_{-K_1}\,|\Delta_{p_1}\,\rangle}{\langle\,\Delta_{p_3}\,|\,V_{\Delta_{p_2}}(1)\,|\Delta_{p_1}\,\rangle}\right) \leqslant \sum_{j=1}^{3} 2\#M_j + \#K_j$$

for $\sum\limits_{j=1}^{3}\#K_j \in 2\mathbb{N}$ while

$$\deg_{\alpha_i}\left(\frac{\langle\,\Delta_{p_3}\,|\,G^\dagger_{-K_3}L^\dagger_{-M_3}\mathcal{L}_{-M_2}\mathcal{G}_{-K_2}V_{\Delta_{p_2}}(1)L_{-M_1}G_{-K_1}\,|\Delta_{p_1}\,\rangle}{\langle\,\Delta_{p_3}\,|\,*V_{\Delta_{p_2}}(1)\,|\Delta_{p_1}\,\rangle}\right) \leqslant \sum_{j=1}^{3} 2\#M_j + \#K_j - 1$$

for $\sum\limits_{j=1}^{3}\#K_j \in 2\mathbb{N}-1$. By the construction of $|p,j\,\rangle_n$ and Proposition 2 one gets:

$$\deg_{\alpha_i}\rho^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) \leqslant j_1^2 + j_2^2 + j_3^2$$

when $j_1 + j_2 + j_3$ is even and

$$\deg_{\alpha_i}\rho^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) \leqslant j_1^2 + j_2^2 + j_3^2 - 1$$

when $j_1 + j_2 + j_3$ is odd.

There are several free field representations of the matrix element ${}_n\langle p_3, j_3|\,V_{p_2,j_2}(1)\,|p_1, j_1\,\rangle_n$. Suppose that all j_i are positive. The state $|p,j\,\rangle_n$ can then be represented either as

$$\Omega(p,j)\chi_{-\frac{2j-1}{2}}\ldots\chi_{-\frac{1}{2}}\,|p\rangle\,,$$

or

$$\Omega(p,j)\chi^{\rm R}_{-\frac{2j-1}{2}}(-p)\ldots\chi^{\rm R}_{-\frac{1}{2}}(-p)\,|-p\rangle\,.$$

There are also two different representations for the descendant field $V_{p,j}(1)$:

$$\frac{\Omega(p,j)}{(2\pi i)^j}\oint\limits_1\frac{d\xi_j}{(\xi_j-1)^j}\cdots\oint\limits_1\frac{d\xi_1}{\xi_1-1}\chi(\xi_j)\ldots\chi(\xi_1)\mathsf{E}^{\alpha}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s, \qquad \alpha = \frac{Q}{2} + ip,$$

and

$$\frac{\Omega(p,j)}{(2\pi i)^j}\oint\limits_1\frac{d\xi_j}{(\xi_j-1)^j}\cdots\oint\limits_1\frac{d\xi_1}{\xi_1-1}\chi^{\rm R}(\xi_j)\ldots\chi^{\rm R}(\xi_1)\mathsf{E}^{Q-\alpha}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s,$$

where

$$\chi(\xi)|_{\mathcal{H}_p\otimes\widetilde{\mathcal{F}}_{\rm NS}} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}(f_k - i\psi_k)\,\xi^{-k-\frac{1}{2}}, \quad \chi^{\rm R}(\xi)|_{\mathcal{H}_p\otimes\widetilde{\mathcal{F}}_{\rm NS}} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}(f_k - i\psi^{\rm R}_k(p))\,\xi^{-k-\frac{1}{2}}$$

are the free field representations of the fields $\tilde\chi(\xi)$, $\tilde\chi^R(\xi)$ (3.6). In consequence there are eight distinct representations of the blow-up factor.

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Let us first consider the one with no reflected fields. Suppose that $p_3 = p_1 - i(\alpha_2 + rb + sb^{-1})$. Then, for $j_1 + j_2 + j_3 \in 2\mathbb{N} \cup \{0\}$:

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3}, \xi_{p_2,j_2}, \xi_{p_1,j_1}|1) = \frac{1}{(2\pi i)^{j_2}} \frac{\Omega(-p_3, j_3)\Omega(p_2, j_2)\Omega(p_1, j_1)}{\langle p_3|\mathsf{E}^{\alpha_2}(1)\mathsf{Q}^r_b\mathsf{Q}^s_{\frac{1}{b}}|p_1\rangle} \tag{3.13}$$

$$\times \oint\limits_1 \frac{d\xi_{j_2}}{(\xi_{j_2}-1)^{j_2}} \cdots \oint\limits_1 \frac{d\xi_1}{\xi_1 - 1} \langle p_3|\chi^\dagger_{-\frac{1}{2}} \cdots \chi^\dagger_{-\frac{2j_3-1}{2}} \chi(\xi_{j_2}) \ldots \chi(\xi_1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}^r_b\mathsf{Q}^s_{\frac{1}{b}}\chi_{-\frac{2j_1-1}{2}} \cdots \chi_{-\frac{1}{2}}|p_1\rangle$$

while for $j_1 + j_2 + j_3 \in 2\mathbb{N} - 1$:

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3}, \xi_{p_2,j_2}, \xi_{p_1,j_1}|1) = \frac{i}{\alpha_2}\frac{1}{(2\pi i)^{j_2}} \frac{\Omega(-p_3, j_3)\Omega(p_2, j_2)\Omega(p_1, j_1)}{\langle p_3|\psi(1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}^r_b\mathsf{Q}^s_{\frac{1}{b}}|p_1\rangle} \tag{3.14}$$

$$\times \oint\limits_1 \frac{d\xi_{j_2}}{(\xi_{j_2}-1)^{j_2}} \cdots \oint\limits_1 \frac{d\xi_1}{\xi_1 - 1} \langle p_3|\chi^\dagger_{-\frac{1}{2}} \cdots \chi^\dagger_{-\frac{2j_3-1}{2}} \chi(\xi_{j_2}) \ldots \chi(\xi_1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}^r_b\mathsf{Q}^s_{\frac{1}{b}}\chi_{-\frac{2j_1-1}{2}} \cdots \chi_{-\frac{1}{2}}|p_1\rangle.$$

The correlator in the integrand of the r.h.s. of (3.13) and of (3.14) can be rewritten as an integral of product of the bosonic

$$\langle 0_b| \otimes \langle\!\langle p_3|\mathsf{E}^{\alpha_2}(1)\mathsf{E}^b(u_1)\ldots\mathsf{E}^b(u_r)\mathsf{E}^{\frac{1}{b}}(u_{r+1})\ldots\mathsf{E}^{\frac{1}{b}}(u_{r+s})|p_1\rangle\!\rangle \otimes |0_b\rangle\,,$$

and the fermionic factor

$$(-1)^{j_3}\,\langle 0|\,\chi(\xi_1)\ldots\chi(\xi_N)\psi(u_1)\ldots\psi(u_{r+s})\,|0\rangle \tag{3.15}$$

with $N = j_1 + j_2 + j_3$, $|0\rangle = |0_f\rangle \otimes |0_{\tilde{f}}\rangle$. From the commutation relations of modes one has

$$\chi(w)\chi(z) \sim 0, \qquad \psi(w)\psi(z) \sim \frac{1}{w-z} \qquad \text{and} \qquad \chi(w)\psi(z) \sim -\frac{i}{w-z}. \tag{3.16}$$

This implies that (3.15) vanishes for $r + s < j_1 + j_2 + j_3$ and therefore the l.h.s. of (3.13) contains a factor

$$\prod_{\substack{0\leqslant r,s \\ r+s\,\in\,2\mathbb{N} \\ r+s<j_1+j_2+j_3}} \left(p_3 - p_1 + i(\alpha_2 + rb + sb^{-1})\right) \quad \propto \quad l^{\mathrm{NS}}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1 + j_2 + j_3\right), \tag{3.17}$$

where $l^{\mathrm{NS}}(x, n)$ has been defined in (2.24) while the l.h.s. of (3.13) is proportional to

$$l^{\mathrm{R}}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1 + j_2 + j_3\right) \tag{3.18}$$

with

$$l^{\mathrm{R}}(x, n) = \prod_{\substack{0\leqslant r,s \\ r+s<n \\ r+s\,\in\,2\mathbb{N}-1}} \left(x + rb + sb^{-1}\right). \tag{3.19}$$

Let us now consider the free-field representations of the blow-up factor containing at least one reflected field. Since the oscillators $\psi^{\mathrm{R}}_k(-p)$ are complicated functions of both the bosonic c_n and the fermionic ψ_k oscillators one cannot factorize the correlators appearing in the corresponding integrands into a bosonic and a fermionic parts. For the same reason the OPE of $\mathsf{E}^{\alpha}(z)$ and $\chi^{\mathrm{R}}(w)$ does not vanish. The free field representations $\psi^{\mathrm{R}}(w)$ and $\psi(z)$ of fields introduced in (3.6) commute with p. Their OPE cannot therefore contain any primary field of the form $\mathsf{E}^{\alpha}(z)$ with $\alpha \neq 0$. The only admissible operators are thus even members of the conformal family of the identity field. Since both $\psi^{\mathrm{R}}(w)$ and $\psi(z)$ are primary fields with conformal dimension $\frac{1}{2}$, one gets

$$\chi^{\mathrm{R}}(w)\chi(z)|_{\mathcal{H}_p\otimes\widetilde{\mathcal{F}}_{\mathrm{NS}}} \sim \frac{C(p)}{w-z},$$

where $C(p) = -\langle\, p, -1 \mid p, 1\rangle \neq 0$.

Taking into account these properties of the reflected field $\psi^{\mathrm{R}}(w)$ one can still conclude that the free field representation must vanish whenever the number of χ field insertions exceeds the number of reflected fields and screening charges. It follows in particular that representation with three reflected fields does not provide any direct information about possible zeros of the blow up factor. The remaining six representations involve the following quotients

$$\frac{\langle p_3|\chi_{\frac{1}{2}}\cdots\chi_{\frac{2j_3-1}{2}}\chi(\xi_{j_2})\cdots\chi(\xi_1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi^{\mathrm{R}}_{-\frac{2j_1-1}{2}}\cdots\chi^{\mathrm{R}}_{-\frac{1}{2}}|-p_1\rangle}{\langle p_3|\_\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|-p_1\rangle} \tag{3.20}$$

for $p_3 = -p_1 - i\left(\alpha_2 + rb + sb^{-1}\right)$, where $\_\mathsf{E}^{\alpha_2}(1)$ denotes either $\mathsf{E}^{\alpha_2}(1)$ (when $j_1+j_2+j_3$ is even) or $*\mathsf{E}^{\alpha_2}(1)$ (when $j_1+j_2+j_3$ is odd),

$$\frac{\langle p_3|\chi_{\frac{1}{2}}\cdots\chi_{\frac{2j_3-1}{2}}\chi^{\mathrm{R}}(\xi_{j_2})\cdots\chi^{\mathrm{R}}(\xi_1)\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi_{-\frac{2j_1-1}{2}}\cdots\chi_{-\frac{1}{2}}|p_1\rangle}{\langle p_3|\_\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|p_1\rangle} \tag{3.21}$$

for $p_3 = p_1 - i\left(Q - \alpha_2 + rb + sb^{-1}\right)$,

$$\frac{\langle -p_3|\chi^{\mathrm{R}}_{\frac{1}{2}}\cdots\chi^{\mathrm{R}}_{\frac{2j_3-1}{2}}\chi(\xi_{j_2})\cdots\chi(\xi_1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi_{-\frac{2j_1-1}{2}}\cdots\chi_{-\frac{1}{2}}|p_1\rangle}{\langle -p_3|\_\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|p_1\rangle} \tag{3.22}$$

for $-p_3 = p_1 - i\left(\alpha_2 + rb + sb^{-1}\right)$,

$$\frac{\langle p_3|\chi_{\frac{2j_3-1}{2}}\cdots\chi_{\frac{1}{2}}\chi^{\mathrm{R}}(\xi_{j_2})\cdots\chi^{\mathrm{R}}(\xi_1)\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi^{\mathrm{R}}_{-\frac{2j_1-1}{2}}\cdots\chi^{\mathrm{R}}_{-\frac{1}{2}}|-p_1\rangle}{\langle p_3|\_\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|-p_1\rangle} \tag{3.23}$$

for $p_3 = -p_1 - i\left(Q - \alpha_2 + rb + sb^{-1}\right)$,

$$\frac{\langle -p_3|\chi^{\mathrm{R}}_{\frac{1}{2}}\cdots\chi^{\mathrm{R}}_{\frac{2j_3-1}{2}}\chi(\xi_{j_2})\cdots\chi(\xi_1)\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi^{\mathrm{R}}_{-\frac{2j_1-1}{2}}\cdots\chi^{\mathrm{R}}_{-\frac{1}{2}}|-p_1\rangle}{\langle -p_3|\_\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|-p_1\rangle} \tag{3.24}$$

for $-p_3 = -p_1 - i(\alpha_2 + rb + sb^{-1})$ and

$$\frac{\langle -p_3|\chi^{\mathrm{R}}_{\frac{1}{2}}\cdots\chi^{\mathrm{R}}_{\frac{2j_3-1}{2}}\chi^{\mathrm{R}}(\xi_{j_2})\cdots\chi^{\mathrm{R}}(\xi_1)\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s\chi_{-\frac{2j_1-1}{2}}\cdots\chi_{-\frac{1}{2}}|p_1\rangle}{\langle -p_3|\_\mathsf{E}^{Q-\alpha_2}(1)\mathsf{Q}_b^r\mathsf{Q}_{\frac{1}{b}}^s|p_1\rangle} \tag{3.25}$$

for $-p_3 = p_1 - i\left(Q - \alpha_2 + rb + sb^{-1}\right)$. They imply the factors

$$l^{\sharp}\left(\frac{1}{2}Q - ip_1 + ip_2 - ip_3, j_2 + j_3 - j_1\right), \tag{3.26}$$

$$l^{\sharp}\left(\frac{1}{2}Q + ip_1 - ip_2 - ip_3, j_1 + j_3 - j_2\right), \tag{3.27}$$

$$l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 + ip_3, j_1 + j_2 - j_3\right), \tag{3.28}$$

$$l^{\sharp}\left(\frac{1}{2}Q - ip_1 - ip_2 - ip_3, j_3 - j_1 - j_2\right), \tag{3.29}$$

$$l^{\sharp}\left(\frac{1}{2}Q - ip_1 + ip_2 + ip_3, j_2 - j_1 - j_3\right), \tag{3.30}$$

$$l^{\sharp}\left(\frac{1}{2}Q + ip_1 - ip_2 + ip_3, j_1 - j_2 - j_3\right), \tag{3.31}$$

respectively, where $\sharp = \mathrm{NS}$ for even $j_1 + j_2 + j_3$, while $\sharp = \mathrm{R}$ for $j_1 + j_2 + j_3$ being odd.

For a given set of positive j_1, j_2, j_3 not all the representations contribute. For instance if the inequalities

$$j_2 + j_3 > j_1, \qquad j_3 + j_1 > j_2, \qquad j_1 + j_2 > j_3, \tag{3.32}$$

hold, (3.17) (resp. (3.18)) and (3.26)–(3.28) are the only products with non-empty ranges of integers r, s. Since the number of pairs (r, s) satisfying $0 \leqslant r + s < n \in 2\mathbb{N}$ with $r + s \in 2\mathbb{Z}$ is equal to $\frac{1}{4}n^2$, the total number of factors in (3.17) and (3.26)–(3.28) in the "even" case is equal to

$$\frac{1}{4}(j_1 + j_2 + j_3)^2 + \frac{1}{4}(j_2 + j_3 - j_1)^2 + \frac{1}{4}(j_1 + j_2 - j_3)^2 + \frac{1}{4}(j_1 + j_3 - j_2)^2 = j_1^2 + j_2^2 + j_3^2.$$

The product of terms (3.17) and (3.26)–(3.28) exhausts therefore the dependence of blow-up factor (3.12) on all α_i. Hence, if inequalities (3.32) are satisfied,

$$\begin{aligned}\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3}, \xi_{p_2,j_2}, \xi_{p_1,j_1}|1) &= C^{j_3}_{j_2 j_1}(b) \\ &\times l^{\mathrm{NS}}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1 + j_2 + j_3\right) l^{\mathrm{NS}}\left(\frac{1}{2}Q - ip_1 + ip_2 - ip_3, j_2 + j_3 - j_1\right) \\ &\times l^{\mathrm{NS}}\left(\frac{1}{2}Q + ip_1 - ip_2 - ip_3, j_1 + j_3 - j_2\right) l^{\mathrm{NS}}\left(\frac{1}{2}Q + ip_1 + ip_2 + ip_3, j_1 + j_2 - j_3\right)\end{aligned} \tag{3.33}$$

for $j_1 + j_2 + j_3 \in 2\mathbb{N}$.

Similarly, since the number of pairs (r, s) satisfying $0 \leqslant r + s < n \in 2\mathbb{N} - 1$ with $r, s \in 2\mathbb{Z} - 1$ is equal to $\frac{1}{4}(n^2 - 1)$, and

$$\begin{aligned}&\frac{1}{4}\left((j_1 + j_2 + j_3)^2 - 1\right) + \frac{1}{4}\left((j_2 + j_3 - j_1)^2 - 1\right) \\ &+ \frac{1}{4}\left((j_1 + j_3 - j_2)^2 - 1\right) + \frac{1}{4}\left((j_1 + j_2 - j_3)^2 - 1\right) = j_1^2 + j_2^2 + j_3^2 - 1\end{aligned}$$

one gets that, if inequalities (3.32) are satisfied,

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = C^{j_3}_{j_2j_1}(b) \tag{3.34}$$
$$\times\, l^{\mathrm{R}}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1+j_2+j_3\right) l^{\mathrm{R}}\left(\frac{1}{2}Q - ip_1 + ip_2 - ip_3, j_2+j_3-j_1\right)$$
$$\times\, l^{\mathrm{R}}\left(\frac{1}{2}Q + ip_1 - ip_2 - ip_3, j_1+j_3-j_2\right) l^{\mathrm{R}}\left(\frac{1}{2}Q + ip_1 + ip_2 + ip_3, j_1+j_2-j_3\right)$$

for $j_1+j_2+j_3 \in 2\mathbb{N}-1$.

Looking for non-empty ranges of products and counting the degree of the resulting polynomial one obtains formulae for the blow up factor for other ranges of positive integers j_1, j_2, j_3:

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = C^{j_3}_{j_2j_1}(b) \tag{3.35}$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1+j_2+j_3\right) l^{\sharp}\left(\frac{1}{2}Q - ip_1 + ip_2 - ip_3, j_2+j_3-j_1\right)$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q - ip_1 + ip_2 + ip_3, j_2-j_1-j_3\right) l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 + ip_3, j_1+j_2-j_3\right),$$

for $j_2 > j_1 + j_3$ (and, consequently $j_1 + j_2 > j_3$ and $j_2 + j_3 > j_1$),

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = C^{j_3}_{j_2j_1}(b) \tag{3.36}$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1+j_2+j_3\right) l^{\sharp}\left(\frac{1}{2}Q + ip_1 - ip_2 + ip_3, j_1-j_2-j_3\right)$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q + ip_1 - ip_2 - ip_3, j_1+j_3-j_2\right) l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 + ip_3, j_1+j_2-j_3\right),$$

for $j_1 > j_2 + j_3$ and

$$\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = C^{j_3}_{j_2j_1}(b) \tag{3.37}$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q + ip_1 + ip_2 - ip_3, j_1+j_2+j_3\right) l^{\sharp}\left(\frac{1}{2}Q - ip_1 + ip_2 - ip_3, j_2+j_3-j_1\right)$$
$$\times\, l^{\sharp}\left(\frac{1}{2}Q + ip_1 - ip_2 - ip_3, j_1+j_3-j_2\right) l^{\sharp}\left(\frac{1}{2}Q - ip_1 - ip_2 - ip_3, j_3-j_1-j_2\right),$$

for $j_3 > j_1 + j_2$.

In order to determine the normalization constant $C^{j_3}_{j_2j_1}(b)$ we assume inequalities (3.32) and use the representations (3.13) and (3.14) in the simplest non-vanishing case , i.e. for $s = 0,\ r = j_1 + j_2 + j_3$ and

$$ip_3 = ip_1 + \alpha_2 + Nb, \qquad N = j_1 + j_2 + j_3.$$

In this case the second line of (3.13) and (3.14) takes the form

$$\begin{aligned} I_{\rm num}\left[\begin{smallmatrix} j_3 \\ j_2 j_1 \end{smallmatrix}\right] &= (-1)^{j_3}\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{N-1}} dt_N \, \langle p_3 |\, \mathsf{E}^{\alpha_2}(1)\mathsf{E}^b(t_1)\ldots\mathsf{E}^b(t_N)\,|p_1\rangle\, h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \\ &= (-1)^{j_3}\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{N-1}} dt_N \prod_{k=1}^N t_k^{-b\alpha_1}(1-t_k)^{-b\alpha_2}\prod_{1\leqslant k<l\leqslant N}(t_k-t_l)^{-b^2} h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \end{aligned}$$

where

$$\begin{aligned} h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) &= \frac{1}{(2\pi i)^{j_2}}\oint_1 \frac{d\xi_{j_2}}{(\xi_{j_2}-1)^{j_2}}\ldots\oint_1\frac{d\xi_1}{\xi_1-1}\; g^{j_3}_{j_2j_1}(t_1,\ldots,t_N), \\ g^{j_3}_{j_2j_1}(t_1,\ldots,t_N) &= \langle 0|\chi_{\frac12}\ldots\chi_{\frac{2j_3-1}{2}}\chi(\xi_{j_2})\ldots\chi(\xi_1)\psi(t_1)\ldots\psi(t_N)\chi_{-\frac{2j_1-1}{2}}\ldots\chi_{-\frac12}|0\rangle. \end{aligned}$$

The integral $h^{j_3}_{j_2j_1}(t_1,\ldots,t_N)$ is by definition totally antisymmetric in t_k variables. The integrand $g^{j_3}_{j_2j_1}$ can be calculated by means of the Wick theorem with a help of OPE-s (3.16). Since all the modes of the field χ anticommute with each other, the only contributions arise from the anticommutators between a mode of the field χ and a mode of the field ψ. Indeed, had we taken into account a contribution from a "pairing" between the fields ψ, we would have been left with a correlator containing only modes of the field χ, which vanishes. This shows that

$$h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \to 0 \quad \text{for} \quad t_k\to t_l,\; 1\leqslant k,l\leqslant N \tag{3.38}$$

and the only possible singularities may arise at $t_l = 1, 0$, $l = 1,\ldots,N$. Since

$$\frac{1}{2\pi i}\oint_1 \frac{d\xi_{j_2}}{(\xi_{j_2}-1)^{j_2}}\psi(t_l)\chi(\xi_{j_2}) \sim -i\frac{1}{2\pi i}\oint_1\frac{d\xi_{j_2}}{(\xi_{j_2}-1)^{j_2}}\frac{1}{t_l-\xi_{j_2}} = -\frac{i}{(t_l-1)^{j_2}}$$

one has

$$h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \sim (t_l-1)^{-j_2} \quad \text{for} \quad t_l\to 1,\; l=1,\ldots,N. \tag{3.39}$$

The commutation relation $\{\psi(t_l),\chi_k\} = -i\, t_l^{k-\frac12}$ implies

$$h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \sim t_l^{-j_1} \quad \text{for} \quad t_l\to 0,\;\; l=1,\ldots,N, \tag{3.40}$$

$$h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) \sim t_l^{j_3-1} \quad \text{for} \quad t_l\to \infty,\; l=1,\ldots,N. \tag{3.41}$$

The only totally antisymmetric function with zeros (3.38), singularities (3.39), (3.40) and asymptotics (3.41) reads

$$h^{j_3}_{j_2j_1}(t_1,\ldots,t_N) = \sigma^{j_3}_{j_2j_1}\prod_{k=1}^N t_k^{-j_1}(1-t_l)^{-j_2}\prod_{1\leqslant k<l\leqslant N}(t_k-t_l)$$

where $\sigma^{j_3}_{j_2j_1}$ does not depend on the variables t_k.

Taking into account sign factors we get from the definition of $h^{j_3}_{j_2j_1}(t_1,\dots,t_N)$ that, up to sub-leading terms

$$h^{j_3}_{j_2j_1}(t_1,\dots,t_N)=\begin{cases} -it_N^{-j_1}h^{j_2}_{j_3\,j_1-1}(t_1,\dots,t_{N-1}) & \text{for}\;\; t_N\to 0,\\ -i(-1)^{N+j_2-1}(t_N-1)^{-j_2}h^{j_2-1}_{j_3\,j_1}(t_1,\dots,t_{N-1}) & \text{for}\;\; t_N\to 1,\\ -i(-1)^{N+j_2-1}t_N^{j_3-1}h^{j_2}_{j_3-1\,j_1}(t_1,\dots,t_{N-1}) & \text{for}\;\; t_N\to\infty.\end{cases}$$

Comparing this formula with the one above we get a recurrence relation uniquely determining $\sigma^{j_3}_{j_2j_1}$. This yields

$$h^{j_3}_{j_2j_1}(t_1,\dots,t_N)=(-i)^N\prod_{k=1}^{N}t_k^{-j_1}(1-t_l)^{-j_2}\prod_{1\leqslant k<l\leqslant N}(t_k-t_l). \tag{3.42}$$

We conclude from (3.42) that $I_{\text{num}}\left[\begin{smallmatrix}j_3\\ j_2j_1\end{smallmatrix}\right]$ is expressed through a standard Selberg integral and thus can be calculated with the result

$$\begin{aligned} I_{\text{num}}\left[\begin{smallmatrix}j_3\\ j_2j_1\end{smallmatrix}\right] &= \\ &=\frac{(-1)^{j_3}}{N!}\prod_{k=0}^{N-1}\frac{\Gamma(1+(k+1)(g+1))\Gamma(1-b\alpha_1-j_1+k(g+1))\Gamma(1-b\alpha_2-j_2+k(g+1))}{\Gamma(g+1)\Gamma\big(2-b\alpha_1-j_1-b\alpha_2-j_2+(N-1+k)(g+1)\big)}, \end{aligned} \tag{3.43}$$

where $g=-\frac{1}{2}b\,Q$, $N=j_1+j_2+j_3$.

Suppose that $j_1+j_2+j_3$ is even. In the special case under consideration the matrix element $\langle p_3|\,\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^N\,|p_1\rangle$ takes the form

$$\begin{aligned} \langle p_3|\,\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^{2m}\,|p_1\rangle &= \int_0^1 dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{2m-1}}dt_{2m}\;\langle p_3|\,\mathsf{E}^{\alpha_2}(1)\prod_{k=1}^{2m}\mathsf{E}^b(t_k)\,|p_3\rangle\,\langle 0_f|\prod_{k=1}^{2m}\psi(t_k)\,|0_f\rangle \\ &=\int_0^1 dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{2m-1}}dt_{2m}\;\prod_{k=1}^{2m}t_k^{-b\alpha_1}(1-t_k)^{-b\alpha_2}\prod_{1\leqslant k<l\leqslant 2n}(t_k-t_l)^{-b^2-1}\,\boldsymbol{P}_{2m}(t_1,\dots,t_{2m}) \end{aligned} \tag{3.44}$$

where

$$\boldsymbol{P}_{2m}(t_1,\dots,t_{2m})=\langle 0_f|\prod_{k=1}^{2m}\psi(t_k)\,|0_f\rangle\prod_{1\leqslant k<l\leqslant 2m}(t_k-t_l)=\operatorname{Pf}\left(\frac{1}{t_k-t_l}\right)\prod_{1\leqslant k<l\leqslant 2m}(t_k-t_l)$$

is a totally symmetric polynomial. We calculate this generalization of the Selberg integral in appendix B. Using (B.27) one obtains

$$\begin{aligned} &\langle p_3|\,\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^{2m}\,|p_1\rangle = \\ &\frac{1}{2^m}\prod_{q=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+q+(2q+j)g)\Gamma(1+q-b\alpha_1+(2q+j-1)g)\Gamma(1+q-b\alpha_2+(2q+j-1)g)}{\Gamma(1+g)\Gamma\big(1+m+q-b(\alpha_1+\alpha_2)+(2(m+q)+j-2)g\big)}. \end{aligned} \tag{3.45}$$

The properties of the Euler gamma function allow to present the ratio of (3.43) and (3.45) in the form

$$\begin{aligned}
&(-1)^{j_3} 2^m \left(\frac{b}{2}\right)^{m^2} l^{\rm NS}(-2mb, j_1+j_2+j_3)\\
&\times\ (-1)^{j_1^2}\left(\frac{b}{2}\right)^{m(m-2j_1)} \frac{l^{\rm NS}(-2p_1-2mb, j_2+j_3-j_1)}{l^{\rm NS}(Q+2ip_1, 2j_1)}\\
&\times\ (-1)^{j_2^2}\left(\frac{b}{2}\right)^{m(m-2j_2)} \frac{l^{\rm NS}(-2ip_2-2mb, j_1+j_3-j_2)}{l^{\rm NS}(Q+2ip_2, 2j_2)}\\
&\times\ (-1)^{(m-j_3)^2}\left(\frac{b}{2}\right)^{n(n-2j_3)} \frac{l^{\rm NS}(Q+2ip_1+2ip_2+2mb, j_1+j_2-j_3)}{l^{\rm NS}(-2ip_1-2ip_2-2mb, 2j_3)},
\end{aligned} \tag{3.46}$$

which completes the calculation of representation (3.13) of the l.h.s. of (3.33). The r.h.s. of (3.33) takes the form

$$\begin{aligned}
C^{j_3}_{j_2j_1}(b)\ &l^{\rm NS}(-2mb, j_1+j_2+j_3)\\
&\times l^{\rm NS}(-2p_1-2mb, j_2+j_3-j_1)\\
&\times l^{\rm NS}(-2ip_2-2mb, j_1+j_3-j_2)\\
&\times l^{\rm NS}(Q+2ip_1+2ip_2+2mb, j_1+j_2-j_3).
\end{aligned} \tag{3.47}$$

Comparing (3.46) with (3.47) we get:

$$C^{j_3}_{j_2j_1}(b) = (-1)^{\frac{j_1+j_2-j_3}{2}} 2^{\frac{j_1+j_2+j_3}{2}}. \tag{3.48}$$

To check this result in the case of odd $j_1+j_2+j_3 = 2m-1$ we observe that the matrix element

$$\langle p_3|\,\psi(1)\mathsf{E}^Q(1)\mathsf{Q}_b^{2m-1}\,|p_1\rangle\ = \tag{3.49}$$

$$= \int\limits_0^1 dt_{2m-1}\int\limits_0^{t_{2m-1}} dt_{2m-2}\ldots\int\limits_0^{t_2} dt_1 \prod_{k=1}^{2m-1} t_k^{-b\alpha_1}(1-t_k)^{-bQ}\prod_{1\leqslant k<l\leqslant 2m-1}(t_l-t_k)^{2g}\,\boldsymbol{P}^{(1)}_{2m-1}(t_{2m-1},\ldots,t_1),$$

where

$$\boldsymbol{P}^{(1)}_{2m-1}(t_{2m-1},\ldots,t_1) = \prod_{1\leqslant k<l\leqslant 2m-1}(t_l-t_k)\,\langle 0_f|\psi(1)\psi(t_{2m-1})\ldots\psi(t_1)|0_f\rangle,$$

can be calculated as a particular limit of integral (3.44). Let us rewrite (3.44) in the form

$$\langle p_3|\,\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^{2m}\,|p_1\rangle =$$

$$= \int\limits_0^1 dt_{2m}\int\limits_0^{t_{2m}} dt_{2m-1}\ldots\int\limits_0^{t_2} dt_1 \prod_{k=1}^{2m} t_k^{-b\alpha_1}(1-t_k)^{-b\alpha_2}\prod_{1\leqslant k<l\leqslant 2m}(t_l-t_k)^{2g}\,\boldsymbol{P}_{2m}(t_{2m},\ldots,t_1).$$

For $t_{2m}\to 1$:

$$\prod_{k=1}^{2m} t_k^{-b\alpha_1}(1-t_k)^{-b\alpha_2}\prod_{1\leqslant k<l\leqslant 2m}(t_l-t_k)^{2g}\,\boldsymbol{P}_{2m}(t_{2m},\ldots,t_1)$$

$$= (1-t_{2m})^{-b\alpha_2}\prod_{k=1}^{2m-1} t_k^{-b\alpha_1}(1-t_k)^{-b\alpha_2+2g+1}\prod_{1\leqslant k<l\leqslant 2m-1}(t_l-t_k)^{2g}\,\boldsymbol{P}^{(1)}_{2m-1}(t_{2m-1},\ldots,t_1)+\ldots$$

where the dots stand for terms sub-leading for $t_{2m}\to 1$.

Using the identity

$$\lim_{a\to -1_+}(1+a)\int_0^1 (1-t)^a h(t)\,dt = h(1),$$

valid for a function $h(t)$ left continuous at $t=1$, we thus get

$$\lim_{\alpha_2\to 1/b}(1-b\alpha_2)\,\langle p_3|\,\mathsf{E}^{\alpha_2}(1)\mathsf{Q}_b^{2m}\,|p_1\rangle = \langle p_3|\,\psi(1)\mathsf{E}^{Q}(1)\mathsf{Q}_b^{2m-1}\,|p_1\rangle\,.$$

Computing the limit one obtains

$$\begin{aligned}\langle p_3|\,\psi(1)\mathsf{E}^{Q}(1)\mathsf{Q}_b^{2m-1}\,|p_1\rangle = \frac{\Gamma(g)}{2^m}\prod_{q=1}^{m-1}\prod_{j=1}^{2}\Gamma\big(q+(2q+j-1)g\big) \\ \times\prod_{q=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma\big(1+q+(2q+j)g\big)\Gamma\big(1+q-b\alpha_1+(2q+j-1)g\big)}{\Gamma(1+g)\Gamma\big(m+q-b\alpha_1+(2(m+q)+j-2)g\big)}.\end{aligned}\tag{3.50}$$

Comparing the ratio of (3.43) and (3.50) with formula (3.34), calculated at $ip_2=\frac{Q}{2}$ and $ip_3=ip_1+Q+(2m-1)b$, one can verify formula (3.48) in the case when $j_1+j_2+j_3$ is odd.

Let us note that the cases when some or all j indices are negative can be obtained either by direct analysis or by changing the sign of corresponding momenta. The special cases when one or more j indices go to zero can be derived using exactly the same method. They can be seen as limiting cases of the situations considered above. For instance in the case of $j_1=j_3, j_2=0$ one gets the formula for the scalar product of normalized states

$${}_n\langle p,-j|p,j\rangle_n = {}_n\langle -p,j|p,j\rangle_n = 2^j l^{\rm NS}(2ip,2j)l^{\rm NS}(Q+2ip,2j).\tag{3.51}$$

In section 4 we shall need 3-point conformal blocks related to the 3-point functions

$$\gamma^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = \begin{cases}\dfrac{\langle V_{p_3,j_3}(\infty)V_{p_2,j_2}(1)V_{p_1,j_1}(0)\rangle}{\langle V_{p_3,0}(\infty)V_{p_2,0}(1)V_{p_1,0}(0)\rangle} & \text{for } j_1+j_2+j_3\in 2\mathbb{Z},\\[2ex] \dfrac{\langle V_{p_3,j_3}(\infty)V_{p_2,j_2}(1)V_{p_1,j_1}(0)\rangle}{\langle V_{p_3,0}(\infty)*V_{p_2,0}(1)V_{p_1,0}(0)\rangle} & \text{for } j_1+j_2+j_3\in 2\mathbb{Z}+1.\end{cases}\tag{3.52}$$

They can be expressed in terms of the 3-point block related to the matrix elements of the vertex operators (3.12) as follows

$$\gamma^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = (-1)^{j_3}\rho^{\rm A}_{\rm NS}(\xi_{-p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1).$$

In the case when all j are positive and inequalities (3.32) are satisfied, using (3.33) and (3.48) one gets for instance

$$\gamma^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1) = (-2)^{\frac{j_1+j_2+j_3}{2}}\begin{cases}\mathcal{B}^{\rm NS}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1) & \text{for } j_1+j_2+j_3\in 2\mathbb{Z},\\[1ex] \mathcal{B}^{\rm R}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1) & \text{for } j_1+j_2+j_3\in 2\mathbb{Z}+1,\end{cases}\tag{3.53}$$

where

$$\mathcal{B}^{\sharp}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1) = l^{\sharp}(\alpha_1+\alpha_2+\alpha_3-Q, j_1+j_2+j_3) \\ \times\, l^{\sharp}(\alpha_1+\alpha_2-\alpha_3, j_1+j_2-j_3) \\ \times\, l^{\sharp}(\alpha_1+\alpha_3-\alpha_2, j_1+j_3-j_2) \\ \times\, l^{\sharp}(\alpha_2+\alpha_3-\alpha_1, j_2+j_3-j_1), \qquad \sharp = \mathrm{NS}, \mathrm{R}. \tag{3.54}$$

3.3 4-point conformal blocks

The states on the l.h.s. of (1.1) are organized in terms of $\mathsf{A}_{\rm NS}$ Verma modules $\mathcal{A}_{\Delta_p} = \mathcal{V}_{\Delta_p}\otimes\widetilde{\mathcal{F}}_{\rm NS}$. In order to compare both sides of (1.1) it is convenient to choose a basis in $\mathcal{A}_{\Delta_p}$ consistent with direct sum decomposition (1.3):

$$L^{\rm L}_{-M}L^{\Gamma}_{-N}\,|p,j\,\rangle_n \tag{3.55}$$

where M,N are arbitrary ordered, integer multi-indices and $j\in\mathbb{Z}$. Vertex operators related to the states $|p,j\,\rangle_n$ take the form ($j>0$):

$$\begin{aligned} V_{p,j}(z) &= \Omega(p,j)\oint\limits_z \frac{dw_j}{(w_j-z)^j}\ldots\oint\limits_z\frac{dw_1}{w_1-z}\chi(w_j)\ldots\chi(w_1)V_{p,0}(z),\\ V_{p,-j}(z) &= \Omega(p,-j)\oint\limits_z \frac{dw_j}{(w_j-z)^j}\ldots\oint\limits_z\frac{dw_1}{w_1-z}\chi^{\rm R}(w_j)\ldots\chi^{\rm R}(w_1)V_{p,0}(z). \end{aligned}$$

For all $j\in\mathbb{Z}$ they are primary with respect to the energy momentum tensors

$$T^{\sigma}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L^{\sigma}_n, \qquad \sigma = \mathrm{L},\Gamma. \tag{3.56}$$

Indeed one can check by explicit calculations that the OPE

$$T^{\sigma}(z)V_{p,j}(w) \sim \frac{1}{(z-w)^2}\,\Delta^{\sigma}(p,j)V_{p,j}(w) + \frac{1}{z-w}\left[L^{\sigma}_{-1},V_{p,j}(w)\right], \qquad \sigma=\mathrm{L},\Gamma. \tag{3.57}$$

holds. By standard contour arguments (3.57) implies

$$\left[L^{\sigma}_n, V_{p,j}(z)\right] = (n+1)z^n\Delta^{\sigma}(p,j)V_{p,j}(1) + z^n\left[L^{\sigma}_{-1},V_{p,j}(1)\right]. \tag{3.58}$$

For L^{σ}_0 eigenstates $L^{\sigma}_0\,|\zeta_1\,\rangle = \Delta^{\sigma}_1\,|\zeta_1\,\rangle, L^{\sigma}_0\,|\zeta_3\rangle = \Delta^{\sigma}_3\,|\zeta_3\rangle$, one has in particular

$$\langle\zeta_3|\left[L^{\sigma}_{-1},V_{p_2,j_2}(z)\right]|\zeta_1\,\rangle = \left(\Delta^{\sigma}_3-\Delta^{\sigma}(p_2,j_2)-\Delta^{\sigma}_1\right)\langle\zeta_3|\,V_{p_2,j_2}(z)\,|\zeta_1\,\rangle. \tag{3.59}$$

It follows from (3.57) and (3.59) that calculating the three-point conformal block in basis (3.55) one can use the conformal Ward identities of the corresponding Liouville theories. This yields the relation

$$\rho^{\rm A}_{\rm NS}(L^{\rm L}_{-M_3}L^{\Gamma}_{-N_3}\xi_{p_3,j_3}, L^{\rm L}_{-M_2}L^{\Gamma}_{-N_2}\xi_{p_2,j_2}, L^{\rm L}_{-M_1}L^{\Gamma}_{-N_1}\xi_{p_1,j_1}|z) \\ = \rho_{\rm L}(L_{-M_3}\nu^{\rm L}_{p_3,j_3}, L_{-M_2}\nu^{\rm L}_{p_2,j_2}, L_{-M_1}\nu^{\rm L}_{p_1,j_1}|z) \\ \times\,\rho_{\Gamma}(L_{-N_3}\nu^{\Gamma}_{p_3,j_3}, L_{-N_2}\nu^{\Gamma}_{p_2,j_2}, L_{-N_1}\nu^{\Gamma}_{p_1,j_1}|z) \\ \times\,\rho^{\rm A}_{\rm NS}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|z) \tag{3.60}$$

where ρ_σ is the three-point conformal block for the Virasoro algebra with the central charge c^σ $(\sigma = \textsc{l}, \textsc{r})$.

Relation (3.60) gives rise to relations between higher conformal blocks. As an example we consider the spheric four-point blocks. As in the case of the NS algebra [34] one has four types of the $\mathsf{A}_{\rm NS}$ algebra blocks. For each type there is one even,

$$\mathcal{T}^{1}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = z^{\Delta - \_\Delta_2 - \Delta_1}\left(1 + \sum_{m\in\mathbb{N}} z^m\, T^m_\Delta\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\right),$$

and one odd,

$$\mathcal{T}^{\frac12}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = z^{\Delta - \_\Delta_2 - \Delta_1}\sum_{k\in\mathbb{N}-\frac12} z^k\, T^k_\Delta\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right],$$

conformal block. Although in correlation functions the even and the odd blocks show up separately it is convenient for our present purposes to combine them into a single function

$$\mathcal{T}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = \mathcal{T}^{1}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) + \mathcal{T}^{\frac12}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z).$$

In the formulae above $\_\Delta_i$ stand for the conformal weight Δ_i of the highest weight state ν_i or the conformal weight $*\Delta_i = \Delta_i + \frac12$ of the state $*\nu_p = G_{-\frac12}\nu_i$. An explicit expression for coefficients depends on the basis used in the factorization of the corresponding four-point chiral correlator. For the basis

$$L_{-M}G_{-K}f_{-N}\nu_p$$

all states with nonzero f excitations drop out from the factorization formula and one gets

$$\begin{aligned} T^t_\Delta\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] &= \\ = \sum_{|K|+|M|=|L|+|N|=t} &\rho_{\rm NS}(\nu_4, \_\nu_3, L_{-M}G_{-K}\nu_\Delta|1)\; B^t(\Delta)^{-1}_{MK,NL}\; \rho_{\rm NS}(L_{-N}G_{-L}\nu_\Delta, \_\nu_2, \nu_1|1), \end{aligned} \tag{3.61}$$

where $B^t(\Delta)^{-1}$ is the matrix inverse to Gram matrix (2.16) in the NS Verma module $\mathcal{V}_\Delta$ and $\rho_{\rm NS}$ denotes the three-point $\mathcal{N} = 1$ superconformal block.[3] It follows that all four-point blocks of the algebra $\mathsf{A}_{\rm NS}$ in the tensor product exactly coincide with the $\mathcal{N} = 1$ superconformal blocks in the super-Liouville factor:

$$\mathcal{T}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = \mathcal{F}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = \mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) + \mathcal{F}^{\frac12}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z). \tag{3.62}$$

where on the r.h.s. the notation of [34] was used.

On the other hand one can factorize on basis (3.55). With identification (3.62) this gives relations between superconformal blocks in the NS sector and Virasoro conformal blocks. In order to make this relation precise one has to extend the notion of the four-point Virasoro conformal block to complex intermediate weights. If one insists on the conjugation rules

$$L_n^\dagger = L_{-n}$$

[3] In order to simplify notation we use the same symbol ν_i for the state $|\Delta_i\rangle$ and $|\Delta_i\rangle \otimes |0_{\tilde f}\rangle$.

the only consistent scalar product on $\mathcal{V}_\Delta \oplus \mathcal{V}_{\bar{\Delta}}$ is given by the pairing

$$\langle \bar{\Delta} \,|\, \Delta \rangle = \overline{\langle \Delta \,|\, \bar{\Delta} \rangle} = c, \tag{3.63}$$

where c is a non-vanishing complex number which we set 1 in the following. The Gram matrix takes the off-diagonal form with the complex conjugated blocks

$$\begin{aligned} B^m(\Delta)_{\bar{M},N} &= \langle \bar{\Delta} \,|\, L_{\bar{M}} L_{-N} \,|\, \Delta \rangle, \\ B^m(\bar{\Delta})_{M,\bar{N}} &= \langle \Delta \,|\, L_M L_{-\bar{N}} \,|\, \bar{\Delta} \rangle. \end{aligned}$$

The identity operator on $\mathcal{V}_\Delta \oplus \mathcal{V}_{\bar{\Delta}}$ can be expressed as

$$\mathrm{id} = \sum_{N,\bar{M}} L_{-N} |\, \Delta \rangle B^{-1}(\Delta)_{N,\bar{M}} \langle \bar{\Delta} \,| L_{\bar{M}} + \sum_{N,\bar{M}} L_{-\bar{N}} |\, \bar{\Delta} \rangle B^{-1}(\bar{\Delta})_{\bar{N},M} \langle \Delta \,| L_M .$$

For each conjugate pair of complex weights $\Delta, \bar{\Delta}$ it is then natural to introduce two four-point conformal blocks:

$$\begin{aligned} \mathcal{F}_\Delta \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) &= z^{\Delta-\Delta_2-\Delta_1} \left(1 + \sum_{m\in\mathbb{N}} z^m \, F^m_\Delta \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \right), \\ F^m_\Delta \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] &= \sum_{|N|=|\bar{M}|=m} \rho(\nu_4, \nu_3, L_{-N}\nu_\Delta|1) \; B^t(\Delta)^{-1}_{N,\bar{M}} \; \rho(L_{-\bar{M}}\nu_{\bar{\Delta}}, \nu_2, \nu_1|1) \end{aligned}$$

and its $\bar{\Delta}$ counterpart. With this definition one has

$$\begin{aligned} \mathcal{F}_{\Delta_p} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) &= \sum_{j\in\mathbb{Z}} A_{\Delta(p,j)} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \; \mathcal{F}^{\mathrm{L}}_{\Delta^{\mathrm{L}}(p,j)} \left[\begin{smallmatrix} \Delta^{\mathrm{L}}_3 & \Delta^{\mathrm{L}}_2 \\ \Delta^{\mathrm{L}}_4 & \Delta^{\mathrm{L}}_1 \end{smallmatrix}\right](z) \; \mathcal{F}^{\Gamma}_{\Delta^{\Gamma}(p,j)} \left[\begin{smallmatrix} \Delta^{\Gamma}_3 & \Delta^{\Gamma}_2 \\ \Delta^{\Gamma}_4 & \Delta^{\Gamma}_1 \end{smallmatrix}\right](z) \; z^{\frac{j^2}{2}}, \\ A_{\Delta(p,j)} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] &= \frac{\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_4,0}, \xi_{p_3,0}, \xi_{p,j}|1) \rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p,-j}, \xi_{p_2,0}, \xi_{p_1,0}|1)}{{}_n\langle\, p, -j \,|p, j\,\rangle_n}, \end{aligned}$$

where we use the simplified notation $\Delta^{\mathrm{L}}_i = \Delta^{\mathrm{L}}(p_i, 0)$, $\Delta^{\Gamma}_i = \Delta^{\Gamma}(p_i, 0)$. Formulae for the other types of blocks follow from the relation

$$G_{-\frac{1}{2}} \,|\Delta_p\rangle \otimes |\, 0_{\tilde{f}}\,\rangle = \frac{1}{2} \,|p, 1\rangle_n - \left(ip + \frac{Q}{2} \right) |\Delta_p\rangle \otimes f_{-\frac{1}{2}} |\, 0_{\tilde{f}}\,\rangle. \tag{3.64}$$

For blocks with a single star the f excitations drop out and one has

$$\begin{aligned} \mathcal{F}_{\Delta_p} \left[\begin{smallmatrix} \Delta_3 * \Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right](z) &= \sum_{j\in\mathbb{Z}} A_{\Delta(p,j)} \left[\begin{smallmatrix} \Delta_3 * \Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right] \; \mathcal{F}^{\mathrm{L}}_{\Delta^{\mathrm{L}}(p,j)} \left[\begin{smallmatrix} \Delta^{\mathrm{L}}_3 * \Delta^{\mathrm{L}}_2 \\ \Delta^{\mathrm{L}}_4 \;\; \Delta^{\mathrm{L}}_1 \end{smallmatrix}\right](z) \; \mathcal{F}^{\Gamma}_{\Delta^{\Gamma}(p,j)} \left[\begin{smallmatrix} \Delta^{\Gamma}_3 * \Delta^{\Gamma}_2 \\ \Delta^{\Gamma}_4 \;\; \Delta^{\Gamma}_1 \end{smallmatrix}\right](z) \; z^{\frac{j^2}{2}}, \\ A_{\Delta(p,j)} \left[\begin{smallmatrix} \Delta_3 * \Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right] &= \frac{1}{2} \frac{\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_4,0}, \xi_{p_3,0}, \xi_{p,j}|1) \rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p,-j}, \xi_{p_2,1}, \xi_{p_1,0}|1)}{{}_n\langle\, p, -j \,|p, j\,\rangle_n}, \end{aligned}$$

where $*\Delta^{\mathrm{L}}_i = \Delta^{\mathrm{L}}(p_i, 1)$, $*\Delta^{\Gamma}_i = \Delta^{\Gamma}(p_i, 1)$.

The relations for double star blocks is slightly more complicated. First of all relation (3.64) implies

$$\mathcal{F}_{\Delta_p} \left[\begin{smallmatrix} *\Delta_3 * \Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right](z) = \frac{1}{4} \mathcal{T}_{\Delta_p} \left[\begin{smallmatrix} \Delta(p_3,1) & \Delta(p_2,1) \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) + \frac{2\Delta_p}{1-z} \mathcal{F}_{\Delta_p} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z).$$

Then using (3.60) one gets

$$\begin{aligned}
\mathcal{F}_{\Delta_p}\left[\begin{smallmatrix} {}^*\Delta_3 {}^*\Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right](z) &= \sum_{j\in\mathbb{Z}} A_{\Delta(p,j)}\left[\begin{smallmatrix} {}^*\Delta_3 {}^*\Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right] \mathcal{F}^{\mathrm{L}}_{\Delta^{\mathrm{L}}(p,j)}\left[\begin{smallmatrix} {}^*\Delta_3^{\mathrm{L}} {}^*\Delta_2^{\mathrm{L}} \\ \Delta_4^{\mathrm{L}} \;\; \Delta_1^{\mathrm{L}} \end{smallmatrix}\right](z)\, \mathcal{F}^{\Gamma}_{\Delta^{\Gamma}(p,j)}\left[\begin{smallmatrix} {}^*\Delta_3^{\Gamma} {}^*\Delta_2^{\Gamma} \\ \Delta_4^{\Gamma} \;\; \Delta_1^{\Gamma} \end{smallmatrix}\right](z)\; z^{\frac{j^2}{2}} \\
&+ \frac{2\Delta_p}{1-z}\sum_{j\in\mathbb{Z}} A_{\Delta(p,j)}\left[\begin{smallmatrix} \Delta_3 \;\Delta_2 \\ \Delta_4 \;\Delta_1 \end{smallmatrix}\right] \mathcal{F}^{\mathrm{L}}_{\Delta^{\mathrm{L}}(p,j)}\left[\begin{smallmatrix} \Delta_3^{\mathrm{L}} \;\Delta_2^{\mathrm{L}} \\ \Delta_4^{\mathrm{L}} \;\Delta_1^{\mathrm{L}} \end{smallmatrix}\right](z)\, \mathcal{F}^{\Gamma}_{\Delta^{\Gamma}(p,j)}\left[\begin{smallmatrix} \Delta_3^{\Gamma} \;\Delta_2^{\Gamma} \\ \Delta_4^{\Gamma} \;\Delta_1^{\Gamma} \end{smallmatrix}\right](z)\; z^{\frac{j^2}{2}}, \\
A_{\Delta(p,j)}\left[\begin{smallmatrix} {}^*\Delta_3 {}^*\Delta_2 \\ \Delta_4 \;\; \Delta_1 \end{smallmatrix}\right] &= \frac{1}{4}\frac{\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p_4,0},\xi_{p_3,1},\xi_{p,j}|1)\rho^{\mathrm{A}}_{\mathrm{NS}}(\xi_{p,-j},\xi_{p_2,1},\xi_{p_1,0}|1)}{{}_n\langle\, p,-j\,|p,j\,\rangle_n}.
\end{aligned}$$

4 Equivalence

4.1 Chiral structure constants

It was shown in subsection 3.1 that the map

$$L^{\mathrm{L}}_{-M}L^{\Gamma}_{-N}\,|p,j\,\rangle_n \quad\longrightarrow\quad L^{\mathrm{L}}_{-M}|\,\nu^{\mathrm{L}}_{p,j}\rangle\otimes L^{\Gamma}_{-N}|\,\nu^{\Gamma}_{p,j}\rangle \tag{4.1}$$

is an isomorphism of $\mathsf{Vir}\oplus\mathsf{Vir}$ representations. It has its counterpart for the corresponding chiral operators

$$\mathcal{L}^{\mathrm{L}}_{-M}\mathcal{L}^{\Gamma}_{-N}V_{p,j} \quad\longrightarrow\quad \mathcal{L}^{\mathrm{L}}_{-M}V^{\mathrm{L}}_{p,j}\otimes\mathcal{L}^{\Gamma}_{-N}V^{\Gamma}_{p,j}.$$

It provides an equivalence of the theories if all three point functions of corresponding operators are equal. We have shown in the previous subsection that for $L^{\mathrm{L}}_n, L^{\Gamma}_n$ generators one can use the same Virasoro Ward identities on both sides of the correspondence. It is then enough to check the correlators of the operators $V_{p,j}$. One obviously has the same situation in the right sector.

The following identities are responsible for the relations between the chiral structure constants:[4]

$$\frac{\Gamma_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(\alpha^{\Gamma}+b^{\Gamma})} = D(b)b^{-\frac{b^2}{1-b^2}\frac{\alpha(Q-\alpha)}{4}}\left(\frac{1-b^2}{2}\right)^{-\frac{\alpha(Q-\alpha)}{8}+\frac{1}{4}}\Gamma^{\mathrm{NS}}_b(\alpha)\,, \tag{4.2}$$

$$\frac{\Gamma_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}}+\frac{1}{2}b^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(\alpha^{\Gamma}+\frac{1}{2}(b^{\Gamma})^{-1}+b^{\Gamma})} = D(b)b^{-\frac{b^2}{1-b^2}\frac{\alpha(Q-\alpha)}{4}+\frac{b\alpha}{2-2b^2}-\frac{b^2}{4-4b^2}}\left(\frac{1-b^2}{2}\right)^{-\frac{\alpha(Q-\alpha)}{8}+\frac{1}{8}}\Gamma^{\mathrm{R}}_b(\alpha)\,, \tag{4.3}$$

$$\frac{\Gamma_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}}-\frac{1}{2}b^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(\alpha^{\Gamma}-\frac{1}{2}(b^{\Gamma})^{-1}+b^{\Gamma})} = D(b)b^{-\frac{b^2}{1-b^2}\frac{\alpha(Q-\alpha)}{4}-\frac{b\alpha}{2-2b^2}+\frac{2+b^2}{4-4b^2}}\left(\frac{1-b^2}{2}\right)^{-\frac{\alpha(Q-\alpha)}{8}+\frac{1}{8}}\Gamma^{\mathrm{R}}_b(\alpha)\,, \tag{4.4}$$

$$D(b) = \frac{\sqrt{2\pi}\,\Gamma_{b^{\mathrm{L}}}(Q^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(b^{\Gamma}+(b^{\Gamma})^{-1})\Gamma_b(Q)\,\Gamma_b\left(\frac{Q}{2}\right)}.$$

[4] We give a simple derivation of these identities along with other useful formulae in appendix C.

The first one is the chiral versions of the identity for upsilon functions proposed in [15]. Using the identities above one gets

$$\mathsf{C}^{\mathrm{L}}_{b}(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2,\alpha^{\mathrm{L}}_1)\mathsf{C}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2,\alpha^{\Gamma}_1) = F(b)\mathsf{C}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1), \tag{4.5}$$

$$\bar{\mathsf{C}}^{\mathrm{L}}_{b}(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2,\alpha^{\mathrm{L}}_1)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2,\alpha^{\Gamma}_1) = F(b)\bar{\mathsf{C}}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1), \tag{4.6}$$

$$\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2+\frac{b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2+\frac{1}{2b^{\Gamma}},\alpha^{\Gamma}_1\right) = \frac{2\sqrt{2}}{\sqrt{2(2\alpha_2)(Q-2\alpha_2)}}F(b)\mathsf{D}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1), \tag{4.7}$$

$$\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2+\frac{b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2+\frac{1}{2b^{\Gamma}},\alpha^{\Gamma}_1\right) = \frac{2\sqrt{2}}{\sqrt{2(2\alpha_2)(Q-2\alpha_2)}}F(b)\bar{\mathsf{D}}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1), \tag{4.8}$$

where

$$F(b) = D(b)\, b^{-\frac{b^2}{1-b^2}\frac{Q^2}{4}}\left(\frac{1-b^2}{2}\right)^{-\frac{Q^2}{8}+\frac{1}{4}}.$$

We shall now analyze the three point chiral functions of operators corresponding to arbitrary $|p,j\rangle_n$ states. For clarity of presentation we restrict ourselves to the case when all j and $\bar{j}$ are positive and inequalities (3.32) are satisfied. Let us first consider the even case $j_1+j_2+j_3\ \in\ 2\mathbb{N}$. Using formulae (C.1) and (C.2) of appendix C and properties of Euler gamma functions one gets

$$\begin{aligned}
&\frac{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2,\alpha^{\mathrm{L}}_1)\mathsf{C}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2,\alpha^{\Gamma}_1)}\\
&=\frac{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2,\alpha^{\mathrm{L}}_1)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2,\alpha^{\Gamma}_1)}\\
&=\left(\prod_{k=1}^{3}\frac{(-1)^{\frac{j_k^2}{2}}}{\sqrt{l(2\alpha_k,2j_k)l(2\alpha_k-Q,2j_k)}}\right)\mathcal{B}^{\mathrm{NS}}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1),
\end{aligned}$$

with $\mathcal{B}^{\mathrm{NS}}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1)$ defined in (3.54). Together with (4.5) and (4.6) it yields

$$\begin{aligned}
&\frac{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\mathsf{C}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1)}\\
&=\frac{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\bar{\mathsf{C}}^{\mathrm{NS}}_{b}(\alpha_3,\alpha_2,\alpha_1)}\\
&=\left(\prod_{k=1}^{3}\frac{(-1)^{\frac{j_k^2}{2}}}{\sqrt{l(2\alpha_k,2j_k)l(2\alpha_k-Q,2j_k)}}\right)\mathcal{B}^{\mathrm{NS}}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1)F(b).
\end{aligned}$$

The comparison with formula (3.53) for the conformal block suggests the rescaling

$$\zeta_{p,j}\equiv|p,j\rangle_{nn} = \frac{1}{\sqrt{2^j l(2ip+Q,2j)l(2ip,2j)}}\,|p,j\rangle_n\,. \tag{4.9}$$

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This gives the normalization ${}_{nn}\langle p,-j\ |p,j\rangle_{nn}=1$ which is in line with the natural normalization we have chosen in subsection 3.3 for Virasoro Verma modules over complex weights (3.63). If we assume this scalar product in direct sum (1.3) the maps

$$\begin{aligned} I\ &:\ L^{\mathrm{L}}_{-M}L^{\Gamma}_{-N}\,|\zeta_{p,j}\,\rangle \longrightarrow L^{\mathrm{L}}_{-M}|\,\nu^{\mathrm{L}}_{p,j}\rangle\otimes L^{\Gamma}_{-N}|\,\nu^{\Gamma}_{p,j}\rangle, \\ \bar{I}\ &:\ L^{\mathrm{L}}_{-\bar{M}}L^{\Gamma}_{-\bar{N}}\,|\bar{\zeta}_{p,j}\,\rangle \longrightarrow L^{\mathrm{L}}_{-\bar{M}}|\,\bar{\nu}^{\mathrm{L}}_{p,j}\rangle\otimes L^{\Gamma}_{-\bar{N}}|\,\bar{\nu}^{\Gamma}_{p,j}\rangle, \end{aligned}$$

become unitary isomorphisms.

For the new states one has

$$\begin{aligned} &\frac{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{F(b)\mathsf{C}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1)} \\ &=\frac{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{F(b)\bar{\mathsf{C}}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1)} \\ &=\gamma^{\mathrm{A}}_{\mathrm{NS}}(\zeta_{p_3,j_3},\zeta_{p_2,j_2},\zeta_{p_1,j_1}|1). \end{aligned} \tag{4.10}$$

In the odd case $j_1+j_2+j_3\ \in\ 2\mathbb{N}+1$ one obtains

$$\begin{aligned} &\frac{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2+\frac{b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2+\frac{1}{2b^{\Gamma}},\alpha^{\Gamma}_1\right)} \\ &=\frac{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3,\alpha^{\mathrm{L}}_2+\frac{b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3,\alpha^{\Gamma}_2+\frac{1}{2b^{\Gamma}},\alpha^{\Gamma}_1\right)} \\ &=\left(\prod_{k=1}^{3}\frac{(-1)^{\frac{j_k^2}{2}}}{\sqrt{l(2\alpha_k,2j_k)l(2\alpha_k-Q,2j_k)}}\right)\sqrt{2\alpha_2(Q-2\alpha_2)}\,\mathcal{B}^{\mathrm{R}}_{j_3j_2j_1}\,(\alpha_3,\alpha_2,\alpha_1)\,F(b) \end{aligned}$$

where $\mathcal{B}^{\mathrm{R}}_{j_3j_2j_1}(\alpha_3,\alpha_2,\alpha_1)$ is given by (3.54). The counterpart of (4.10) reads

$$\begin{aligned} &\frac{\mathsf{C}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\mathsf{C}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{2\sqrt{2}F(b)\mathsf{D}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1)} \\ &=\frac{\bar{\mathsf{C}}^{\mathrm{L}}_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}_3+\frac{j_3b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_2+\frac{j_2b^{\mathrm{L}}}{2},\alpha^{\mathrm{L}}_1+\frac{j_1b^{\mathrm{L}}}{2}\right)\bar{\mathsf{C}}^{\Gamma}_{b^{\Gamma}}\left(\alpha^{\Gamma}_3+\frac{j_3}{2b^{\Gamma}},\alpha^{\Gamma}_2+\frac{j_2}{2b^{\Gamma}},\alpha^{\Gamma}_1+\frac{j_1}{2b^{\Gamma}}\right)}{2\sqrt{2}F(b)\bar{\mathsf{D}}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1)} \\ &=\gamma^{\mathrm{A}}_{\mathrm{NS}}(\zeta_{p_3,j_3},\zeta_{p_2,j_2},\zeta_{p_1,j_1}|1). \end{aligned} \tag{4.11}$$

It follows from (4.10) and (4.11) that after an appropriate overall rescaling of the NS chiral structure constants the maps $I,\bar{I}$ provide equivalence of chiral correlators in the left and in the right chiral sector, respectively.

4.2 Correlation functions

We shall now compare the full correlation functions on both sides of the correspondence. If one chooses the standard scalar product on the left-right tensor products

$$\langle \xi \otimes \bar{\xi} \,|\chi \otimes \bar{\chi}\rangle = \langle \xi \,|\chi\rangle \langle \bar{\xi} \,|\bar{\chi}\rangle$$

on both sides of the correspondence, then the map $\mathcal{I} : \mathcal{H}^{\mathrm{SL}} \to \mathcal{H}^{\mathrm{LL}}$,

$$\mathcal{I} \;:\; \zeta_{p,j} \otimes \bar{\zeta}_{p,\bar{j}} \;\longrightarrow\; \begin{cases} \nu^{\mathrm{L}}_{p,j,\bar{j}} \otimes \bar{\nu}^{\Gamma}_{p,j,\bar{j}} & \text{for } j,\bar{j} \in 2\mathbb{Z}, \\ i\,\nu^{\mathrm{L}}_{p,j,\bar{j}} \otimes \bar{\nu}^{\Gamma}_{p,j,\bar{j}} & \text{for } j,\bar{j} \in 2\mathbb{Z}+1, \end{cases}$$

where

$$\nu^{\mathrm{L}}_{p,j,\bar{j}} \equiv \nu^{\mathrm{L}}_{p,j} \otimes \bar{\nu}^{\mathrm{L}}_{p,\bar{j}}, \qquad \nu^{\Gamma}_{p,j,\bar{j}} \equiv \nu^{\Gamma}_{p,j} \otimes \bar{\nu}^{\Gamma}_{p,\bar{j}},$$

is a unitary isomorphism. Its counterpart for the local fields

$$\mathcal{I} \;:\; \Phi_{p,j,\bar{j}} \;\longrightarrow\; \begin{cases} \Phi^{\mathrm{L}}_{p,j,\bar{j}} \otimes \Phi^{\Gamma}_{p,j,\bar{j}} & \text{for } j,\bar{j} \in 2\mathbb{Z}, \\ i\,\Phi^{\mathrm{L}}_{p,j,\bar{j}} \otimes \Phi^{\Gamma}_{p,j,\bar{j}} & \text{for } j,\bar{j} \in 2\mathbb{Z}+1, \end{cases} \tag{4.12}$$

is expected to provide the SL-LL equivalence if all correlation functions of corresponding operators are equal. The choice of phase in (4.12) may seem strange at this stage but it is in fact indispensable as we shall see in the following.

If we restrict ourselves to the correlation functions on the sphere it is enough to show that all 3-point functions coincide and that the factorization procedures are the same. For the first part let us recall that for $L^{\mathrm{L}}_n, L^{\Gamma}_n$ generators one can use the same Virasoro Ward identities on both sides of the correspondence. It is then sufficient to check the 3-point functions of the operators $\Phi_{p,j,\bar{j}}$.

Let us first verify the equivalence for the superprimary fields $\Phi_{p,0,0}$. From definitions (1.4), (1.5), (1.7) and chiral relations (4.5), (4.6) one has[5]

$$\frac{\Big\langle \Phi^{\mathrm{L}}_{p_3,0,0}\Phi^{\mathrm{L}}_{p_2,0,0}\Phi^{\mathrm{L}}_{p_1,0,0}\Big\rangle_{\mathrm{L}} \Big\langle \Phi^{\Gamma}_{p_3,0,0}\Phi^{\Gamma}_{p_2,0,0}\Phi^{\Gamma}_{p_1,0,0}\Big\rangle_{\Gamma}}{\Big\langle \Phi_{p_3,0,0}\Phi_{p_2,0,0}\Phi_{p_1,0,0}\Big\rangle_{\mathrm{SL}}} = F(b)^2\,\frac{M^{\mathrm{L}}_{b^{\mathrm{L}}}M^{\Gamma}_{b^{\Gamma}}}{M^{\mathrm{NS}}_b}.$$

where

$$F(b)^2 = \frac{\Upsilon_{b^{\Gamma}}(b^{\Gamma})\Upsilon_b(b)\,\Upsilon_b\left(\frac{Q}{2}\right)}{\Upsilon_{b^{\mathrm{L}}}(b^{\mathrm{L}})}\, b^{-\frac{b^2}{1-b^2}\frac{Q^2}{2}} \left(\frac{1-b^2}{2}\right)^{-\frac{Q^2}{4}+\frac{1}{2}}.$$

This yields the relative normalization condition (1.11) which we assume is satisfied.

Let us now conisder 3-point function with one excitation. From definition (1.8), normalization (4.9) and relation (3.64) one has on the SL side

$$\Big\langle \Phi_{p_3,0,0}\Phi_{p_2,1,1}\Phi_{p_1,0,0}\Big\rangle_{\mathrm{SL}} = i\frac{8}{2(2\alpha_2)(Q-2\alpha_2)} M^{\mathrm{NS}}_b \mathsf{D}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1)\bar{\mathsf{D}}^{\mathrm{NS}}_b(\alpha_3,\alpha_2,\alpha_1).$$

[5] For the sake of clarity we drop locations of fields which are assumed to be the standard ones $(0,0),(1,1),(\infty,\infty)$.

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The calculations on the LL side can be readily done using chiral formulae (4.7), (4.8). If condition (1.11) is satisfied one gets

$$\frac{i\left\langle \Phi^{\rm L}_{p_3,0,0}\Phi^{\rm L}_{p_2,1,1}\Phi^{\rm L}_{p_1,0,0}\right\rangle_{\rm L}\left\langle \Phi^{\Gamma}_{p_3,0,0}\Phi^{\Gamma}_{p_2,1,1}\Phi^{\Gamma}_{p_1,0,0}\right\rangle_{\Gamma}}{\left\langle \Phi_{p_3,0,0}\Phi_{p_2,1,1}\Phi_{p_1,0,0}\right\rangle_{\rm SL}} = 1.$$

Note that this partially explains our choice of phase in (4.12). Using essentially the same calculations one checks the formula above for the three other cases $(j_2,\bar{j}_2) = (1,-1),(-1,1),(-1,-1)$.

We shall now consider the three point function of operators $\Phi_{p,j,\bar{j}}$ corresponding to arbitrary $\zeta_{p,j}\otimes\zeta_{p,\bar{j}}$ states. For simplicity we restrict ourselves to the case when all j and $\bar{j}$ are positive and inequalities (3.32) are satisfied. Let us first assume that the sum of all left j indices is even[6]

$$j_1+j_2+j_3\ \in\ 2\mathbb{N}.$$

There are two subcases: all left j indices are even or two of them are odd and one is even. In the first subcase one has on the SL side[7]

$$\begin{aligned}&\left\langle \Phi_{p_3,j_3,\bar{j}_3}\Phi_{p_2,j_2,\bar{j}_2}\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\rm SL}\\ &\quad= C^{\rm NS}_b(\alpha_3,\alpha_2,\alpha_1)\gamma^{\rm A}_{\rm NS}(\zeta_{p_3,j_3}\zeta_{p_2,j_2}\zeta_{p_1,j_1}|1)\gamma^{\rm A}_{\rm NS}(\bar\zeta_{p_3,\bar{j}_3}\bar\zeta_{p_2,\bar{j}_2}\bar\zeta_{p_1,\bar{j}_1}|1).\end{aligned}$$

Then by chiral formulae (4.10) one obtains

$$\frac{\left\langle \Phi^{\rm L}_{p_3,j_3,\bar{j}_3}\Phi^{\rm L}_{p_2,j_2,\bar{j}_2}\Phi^{\rm L}_{p_1,j_1,\bar{j}_1}\right\rangle_{\rm L}\left\langle \Phi^{\Gamma}_{p_3,j_3,\bar{j}_3}\Phi^{\Gamma}_{p_2,j_2,\bar{j}_2}\Phi^{\Gamma}_{p_1,j_1,\bar{j}_1}\right\rangle_{\Gamma}}{\left\langle \Phi_{p_3,j_3,\bar{j}_3}\Phi_{p_2,j_2,\bar{j}_2}\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\rm SL}} = 1.$$

In the case of two odd and one even j's one has instead

$$\begin{aligned}&\left\langle \Phi_{p_3,j_3,\bar{j}_3}\Phi_{p_2,j_2,\bar{j}_2}\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\rm SL}\\ &\quad= -C^{\rm NS}_b(\alpha_3,\alpha_2,\alpha_1)\gamma^{\rm A}_{\rm NS}(\zeta_{p_3,j_3}\zeta_{p_2,j_2}\zeta_{p_1,j_1}|1)\gamma^{\rm A}_{\rm NS}(\bar\zeta_{p_3,\bar{j}_3}\bar\zeta_{p_2,\bar{j}_2}\bar\zeta_{p_1,\bar{j}_1}|1)\end{aligned}$$

with the minus sign coming from the anti-commutation of the odd left and the odd right excitations. On the other hand due to our choice of phase in (4.12) one gets on the LL side the factor i^2 so the 3-point functions on both sides coincide.

Let us now turn to the odd case

$$j_1+j_2+j_3\ \in\ 2\mathbb{N}+1.$$

As before it splits into two subcases: one left index is odd and two are even or all of them are odd. In the first case the splitting of the 3-point function into A-algebra conformal blocks does not develop any extra sign:

$$\begin{aligned}&\left\langle \Phi_{p_3,j_3,\bar{j}_3}\Phi_{p_2,j_2,\bar{j}_2}\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\rm SL}\\ &\quad= \tilde{C}^{\rm NS}_b(\alpha_3,\alpha_2,\alpha_1)\gamma^{\rm A}_{\rm NS}(\zeta_{p_3,j_3}\zeta_{p_2,j_2}\zeta_{p_1,j_1}|1)\gamma^{\rm A}_{\rm NS}(\bar\zeta_{p_3,\bar{j}_3}\bar\zeta_{p_2,\bar{j}_2}\bar\zeta_{p_1,\bar{j}_1}|1).\end{aligned}$$

[6]With the GSO projection assumed the right indices satisfy the same condition.

[7]The same left and right chiral Ward identities are assumed so the left and the right 3-point blocks are the same function.

Then by chiral formulae (4.11) one obtains

$$\frac{i\left\langle \Phi^{\mathrm{L}}_{p_3,j_3,\bar{j}_3}\,\Phi^{\mathrm{L}}_{p_2,j_2,\bar{j}_2}\,\Phi^{\mathrm{L}}_{p_1,j_1,\bar{j}_1}\right\rangle_{\mathrm{L}}\left\langle \Phi^{\Gamma}_{p_3,j_3,\bar{j}_3}\,\Phi^{\Gamma}_{p_2,j_2,\bar{j}_2}\,\Phi^{\Gamma}_{p_1,j_1,\bar{j}_1}\right\rangle_{\Gamma}}{\left\langle \Phi_{p_3,j_3,\bar{j}_3}\,\Phi_{p_2,j_2,\bar{j}_2}\,\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\mathrm{SL}}} = 1.$$

In the subcase of all odd indices one has the extra minus sign on the SL side and the extra i^2 factor on the LL side, so again the 3-point functions coincide. All cases involving other inequalities and non-positive indices can be analyzed in the same way. The final result can be compactly written as follows

$$\left\langle \Phi_{p_3,j_3,\bar{j}_3}\,\Phi_{p_2,j_2,\bar{j}_2}\,\Phi_{p_1,j_1,\bar{j}_1}\right\rangle_{\mathrm{SL}} = \left\langle \mathcal{I}(\Phi_{p_3,j_3,\bar{j}_3})\mathcal{I}(\Phi_{p_2,j_2,\bar{j}_2})\mathcal{I}(\Phi_{p_1,j_1,\bar{j}_1})\right\rangle_{\mathrm{LL}}.$$

Let us stress the role the phase i introduced in (4.12) plays in the formula above — it reproduces the factor i in the NS structure constant $\tilde{C}_b^{\mathrm{NS}}$ (1.8) and the extra minus signs coming from the anti-commutation of odd objects on the SL side.

The last step of our proof of the SL-LL equivalence is to compare the factorization of correlation functions. To this end we choose in $\mathcal{H}^{\mathrm{SL}}$ the basis

$$L^{\mathrm{L}}_{-M}L^{\Gamma}_{-N}\,|\zeta_{p,j}\rangle \otimes \bar{L}^{\mathrm{L}}_{-\bar{M}}\bar{L}^{\Gamma}_{-\bar{N}}\,|\bar{\zeta}_{p,\bar{j}}\rangle\,. \tag{4.13}$$

The Virasoro generators in the formula above are all even operators and the map $\mathcal{I}$ is an isomorphisms of $\mathsf{Vir}\oplus\mathsf{Vir}\oplus\bar{\mathsf{Vir}}\oplus\bar{\mathsf{Vir}}$ representations one thus can safely drop them from the subsequent formulae. Let us consider for instance the factorization of the 4-point function on basis (4.13). In the simplified notation it takes the form

$$\begin{aligned}\left\langle \Phi_4\Phi_3\Phi_2\Phi_1\right\rangle_{\mathrm{SL}} = &\int_p \sum_{j,\bar{j}\in 2\mathbb{Z}} \left\langle \Phi_4\Phi_3\Phi_{p,j,\bar{j}}\right\rangle_{\mathrm{SL}}\left\langle \Phi_{-p,j,\bar{j}}\,\Phi_2\Phi_1\right\rangle_{\mathrm{SL}}\\ &-\int_p \sum_{j,\bar{j}\in 2\mathbb{Z}+1} \left\langle \Phi_4\Phi_3\Phi_{p,j,\bar{j}}\right\rangle_{\mathrm{SL}}\left\langle \Phi_{-p,j,\bar{j}}\,\Phi_2\Phi_1\right\rangle_{\mathrm{SL}}.\end{aligned}$$

On the LL side one has

$$\begin{aligned}&\left\langle \mathcal{I}(\Phi_4)\mathcal{I}(\Phi_3)\mathcal{I}(\Phi_2)\mathcal{I}(\Phi_1)\right\rangle_{\mathrm{LL}}\\ &\quad= \int_p \sum_{j,\bar{j}\in 2\mathbb{Z}} \left\langle \mathcal{I}(\Phi_4)\mathcal{I}(\Phi_3)\Phi^{\mathrm{L}}_{p,j,\bar{j}}\otimes\Phi^{\Gamma}_{p,j,\bar{j}}\right\rangle_{\mathrm{LL}}\left\langle \Phi^{\mathrm{L}}_{-p,j,\bar{j}}\otimes\Phi^{\Gamma}_{-p,j,\bar{j}}\,\mathcal{I}(\Phi_2)\mathcal{I}(\Phi_1)\right\rangle_{\mathrm{LL}}\\ &\qquad-\int_p \sum_{j,\bar{j}\in 2\mathbb{Z}+1} \left\langle \mathcal{I}(\Phi_4)\mathcal{I}(\Phi_3)\,i\,\Phi^{\mathrm{L}}_{p,j,\bar{j}}\otimes\Phi^{\Gamma}_{p,j,\bar{j}}\right\rangle_{\mathrm{LL}}\left\langle i\,\Phi^{\mathrm{L}}_{-p,j,\bar{j}}\otimes\Phi^{\Gamma}_{-p,j,\bar{j}}\,\mathcal{I}(\Phi_2)\mathcal{I}(\Phi_1)\right\rangle_{\mathrm{LL}}\\ &\quad= \int_p \sum_{j,\bar{j}\in 2\mathbb{Z}} \left\langle \mathcal{I}(\Phi_4)\mathcal{I}(\Phi_3)\mathcal{I}(\Phi_{p,j,\bar{j}})\right\rangle_{\mathrm{LL}}\left\langle \mathcal{I}(\Phi_{-p,j,\bar{j}})\,\mathcal{I}(\Phi_2)\mathcal{I}(\Phi_1)\right\rangle_{\mathrm{LL}}\\ &\qquad-\int_p \sum_{j,\bar{j}\in 2\mathbb{Z}+1} \left\langle \mathcal{I}(\Phi_4)\mathcal{I}(\Phi_3)\,\mathcal{I}(\Phi_{p,j,\bar{j}})\right\rangle_{\mathrm{LL}}\left\langle \mathcal{I}(\Phi_{-p,j,\bar{j}})\,\mathcal{I}(\Phi_2)\mathcal{I}(\Phi_1)\right\rangle_{\mathrm{LL}}.\end{aligned}$$

One thus gets the exact equivalence of the 4-point functions. It is also clear that the simple mechanism above works for any factorization of any n-point function on the sphere as well.

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A Fermionic state properties

In this appendix we shall prove the propositions of section 2.2.

Proof of Prop. 1. In the coefficients of the decomposition

$$\begin{aligned}\tilde{\psi}_{-L}(p)\,|\Delta_p\rangle &= \sum_{MK} S^n(p)^{-1}_{MK,\emptyset L} L_{-M}G_{-K}\,|\Delta_p\rangle \\ &= \sum_{MK,M'K'} N^n_{NL}\overline{S^n(p)}_{\emptyset,M'K'}\,B^n(p)^{-1}_{M'K',MK}L_{-M}G_{-K}\,|\Delta_p\rangle\end{aligned}$$

the poles arise due to the existence of a singular vector $|\chi_{rs}\rangle \in \mathcal{V}_{\Delta_{rs}}$ on level rs. Let χ^{NL}_{rs} denote the coefficients of $|\chi_{rs}\rangle$ in the basis $L_{-N}G_{-L}\,|\Delta_{rs}\rangle$,

$$\begin{aligned}|\chi_{rs}\rangle &= D_{rs}\,|\Delta_{rs}\rangle\,, \\ D_{rs} &= \sum_{N,L}\chi^{NL}_{rs}L_{-N}G_{-L}\,.\end{aligned} \tag{A.1}$$

We normalize $|\chi_{rs}\rangle$ such that the coefficient at $(L_{-1})^{rs}\,|\Delta_{rs}\rangle$ is equal 1. For $rs < |K|$ consider vectors of the form

$$L_{-N}G_{-L}\,D_{rs}\,|\Delta_p\rangle\,, \qquad |N|+|L| = |K|-rs\,,$$

so that $|\chi_{rs}\rangle = \lim_{p\to p_{rs}} D_{rs}\,|\Delta_p\rangle$. The set of these vectors can be always extended to a full basis in $\mathcal{V}^{|K|}_{\Delta}$. Working in such a basis and using the properties of the Gram matrix $B^{|K|}_{c,\Delta}$ and its inverse [34] one gets

$$\begin{aligned}&\lim_{p\to p_{rs}}(p-p_{rs})\,\tilde{\psi}(p)_{-K}\,|\Delta_p\rangle \\ &\quad= \frac{A_{rs}}{2p_{rs}}\sum\left(\lim_{p\to p_{rs}}\langle\Delta_p|\,D^\dagger_{rs}G^\dagger_{-L}L^\dagger_{-N}\tilde{\psi}(p)_{-K}\,|\Delta_p\rangle\right)\left[B^{|K|-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\right]^{NL,N'L'} L_{-N'}G_{-L'}\,|\chi_{rs}\rangle\,,\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}A_{rs} &= \lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\Delta_{rs}|\,D^\dagger_{rs}D_{rs}\,|\Delta_{rs}\rangle}{\Delta-\Delta_{rs}}\right)^{-1}. \\ &= 2^{rs-2}\prod_{m=1-r}^{r}\prod_{n=1-s}^{s}\left(mb+nb^{-1}\right)^{-1}\end{aligned}$$

and $m+n \in 2\mathbb{Z}$, $(m,n) \neq (0,0),(r,s)$. Using unitary isomorphism (2.16) one easily checks that the limit in (A.2) exists and is finite

$$\lim_{p\to p_{rs}} \langle \Delta_p|\, D^\dagger_{rs} G^\dagger_{-L} L^\dagger_{-N} \tilde{\psi}(p)_{-K}\,|\Delta_p\rangle = \lim_{p\to p_{rs}} \langle p|\, D(p)^\dagger_{rs} G(p)^\dagger_{-L} L(p)^\dagger_{-N} \psi_{-K}\,|p\rangle$$
$$= \langle p_{rs}|\, D(p_{rs})^\dagger_{rs} G(p_{rs})^\dagger_{-L} L(p_{rs})^\dagger_{-N} \psi_{-K}\,|p_{rs}\rangle\,,$$

where $D(p)_{rs} = \sum_{N,L} \chi^{NL}_{rs} L(p)_{-N} G(p)_{-L}$. Hence (A.2) takes the form

$$\lim_{p\to p_{rs}} (p-p_{rs})\,\tilde{\psi}(p)_{-K}\,|\Delta_p\rangle \tag{A.3}$$
$$= \frac{A_{rs}}{2p_{rs}} \sum \langle \chi(p_{rs})|\, G(p_{rs})^\dagger_{-L} L(p_{rs})^\dagger_{-N} \psi_{-K}\,|p_{rs}\rangle\ \left[B^{|K|-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\right]^{NL,N'L'} L_{-N'} G_{-L'}\,|\chi_{rs}\rangle\,,$$

where $\langle \chi(p_{rs})| = \langle p_{rs}|\, D^\dagger_{rs}(p_{rs})$.

We shall show that due to special properties of pure fermionic states some of limits (A.3) vanish. Let us first observe that up to an overall normalization the singular vectors $\langle \chi(p_{rs})|$ can be constructed by means of the screening charges [32]:

$$\langle p_{rs} - irb|\, \mathsf{Q}^r_b \ = \oint\limits_{\infty} dz_r \oint\limits_{\mathcal{C}_{r-1}} dz_{r-1} \ldots \oint\limits_{\mathcal{C}_1} dz_1\ \langle p_0|\, \psi(z_1)\mathsf{E}^b(z_1)\ldots\psi(z_r)\mathsf{E}^b(z_r) \tag{A.4}$$

for $r \leqslant s$ or

$$\left\langle p_{rs} - isb^{-1}\right|\, \mathsf{Q}^s_{\frac{1}{b}} \ = \oint\limits_{\infty} dz_s \oint\limits_{\mathcal{C}_{s-1}} dz_{s-1} \ldots \oint\limits_{\mathcal{C}_1} dz_1\ \langle p_0|\, \psi(z_1)\mathsf{E}^{\frac{1}{b}}(z_1)\ldots\psi(z_s)\mathsf{E}^{\frac{1}{b}}(z_s) \tag{A.5}$$

for $s \leqslant r$, where all the integration contours are closed.[8] It follows that the singular vector $\langle \chi(p_{rs})|$ contains at most $\min\{r,s\}$ fermionic excitations (oscillators).

Suppose that $r \leqslant s$ and consider matrix elements of the form

$$\langle \chi(p_{rs})| G_L L_N \psi_{-\frac{2j-1}{2}} \ldots \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|p_{rs}\rangle$$

where, for notational convenience the argument p_{rs} of generators L_n and G_l is suppressed. Since for all $n>0$:

$$L_n \psi_{-\frac{2j-1}{2}} \ldots \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|p_{rs}\rangle = 0,$$

we can restrict ourselves to matrix elements of the form

$$\langle \chi(p_{rs})| G_{l_1}\ldots G_{l_m} \psi_{-\frac{2j-1}{2}} \ldots \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|p_{rs}\rangle \tag{A.6}$$

where $l_1 < l_2 < \ldots < l_m$ and

$$\frac{rs}{2} + \sum_{i=1}^{m} l_i = \frac{1}{2} j^2. \tag{A.7}$$

Since

$$\{G_l, \psi_k\} = (1-\delta_{l+k}) c_{l+k} + \big(l - iQk\big)\delta_{l+k}$$

[8] This condition restricts momenta to the discrete subset $\{p_{rs}\}$.

the state

$$G_{l_1}\dots G_{l_m}\psi_{-\frac{2j-1}{2}}\dots\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|p_{rs}\rangle$$

contains at least $j-m$ fermionic excitations. On the other hand the state $\langle\chi(p_{rs})|$ contains at most r fermionic excitations. The inequality

$$r \geqslant k-m \tag{A.8}$$

is thus a necessary condition for (A.6) not to vanish. Since all l_i are different

$$\sum_{i=1}^{m} l_i \geqslant \frac{1}{2}\sum_{i=1}^{m}(2i-1) = \frac{1}{2}m^2 \geqslant \frac{1}{2}(j-r)^2$$

where the second inequality follows from (A.8). Using (A.7) we thus get

$$\frac{1}{2}\left(j^2-rs\right) = \sum_{i=1}^{m} l_i \geqslant \frac{1}{2}(j-r)^2$$

or, equivalently

$$r+s \leqslant 2j. \tag{A.9}$$

It follows that all the states at the level $\frac{1}{2}j^2 > \frac{rs}{2}$, generated from $\langle\chi(p_{rs})|$ by the operators G_l and L_n are orthogonal to the state $\psi_{-\frac{2j-1}{2}}\dots\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|p_{rs}\rangle$ *unless* condition (A.9) is satisfied.

Consider now a matrix element of the form

$$\langle\chi(p_{rs})|G_L L_N\psi_{-J'}|p_{rs}\rangle \tag{A.10}$$

where K' is some subset of $J=\{\frac{2j-1}{2},\dots,\frac{3}{2},\frac{1}{2}\}$. Since

$$[L_m,\psi_k] = -\left(\frac{1}{2}m+k\right)\psi_{m+k},$$

the state $L_N\psi_{-K'}|p_{rs}\rangle$ (if non-zero) is a combination of states of the form $\psi_{-K}|p_{rs}\rangle$, where again K is a subset of J. We can thus restrict ourselves to matrix elements of the form

$$\langle\chi(p_{rs})|G_L\psi_{-K}|p_{rs}\rangle. \tag{A.11}$$

Suppose that for $r+s>2k$ there exists a multi-index L such that (A.11) is non-zero. As a polynomial in b and b^{-1} the matrix element (A.11) has the large b leading term

$$\langle\chi(p_{rs})|G_L\psi_{-K}|p_{rs}\rangle \stackrel{b\to\infty}{=} \mathcal{N}b^{n_0}+\mathcal{O}\left(b^{n_0-1}\right), \qquad \mathcal{N}\neq 0.$$

We shall now calculate the large b leading term of the collerator

$$\langle\chi(p_{rs})|G_L G_{J\setminus K}\psi_{-J}|p_{rs}\rangle \tag{A.12}$$

where $J \setminus K$ is the multi-index composed of those indices which complete K to J. To this end it is sufficient to keep only the "linear" parts of the generators

$$G_k(p_{rs}) \sim ib\left(k + \frac{r}{2}\right)\psi_k.$$

This yields

$$\langle\chi(p_{rs})|G_L G_{J\setminus K}\psi_{-K}|p_{rs}\rangle \overset{b\to\infty}{=} \pm i^{\#J-\#K}\prod_{l\in J\setminus K}\left(l+\frac{r}{2}\right)\mathcal{N}\, b^{n_0+\#L-\#J}\left(1+\mathcal{O}\left(b^{-1}\right)\right)$$

and therefore the correlator on the l.h.s. does not vanish in contradiction with our previous considerations. This completes the proof in the general case.

Proof of prop. 2. The aim is to calculate the upper bound for the degree of the coefficients of the decomposition

$$\Omega(p,j)\,\tilde{\psi}(p)_{-K}\,|\Delta_p\rangle = \sum \Omega(p,j)S^n(p)^{-1}_{NL,\emptyset K}\,L_{-N}G_{-L}\,|\Delta_p\rangle$$

where $K \subset J = \{\frac{2j-1}{2},\ldots,\frac{1}{2}\}$. Our strategy is to consider this equation in the free field representation

$$\Omega(p,j)\,\psi_{-K}\,|p\rangle = \sum \Omega(p,j)S^n(p)^{-1}_{NL,\emptyset K}\,L(p)_{-N}G(p)_{-L}\,|p\rangle$$

and express all the objects involved it terms of new variables with well defined scaling properties. Let ϵ_1,ϵ_2 be real positive parameters such that $b=\sqrt{\frac{\epsilon_1}{\epsilon_2}}$. We introduce the rescaled oscillators and the momentum

$$d_n = \sqrt{\epsilon_1\epsilon_2}c_n, \quad \mathsf{q} = \sqrt{\epsilon_1\epsilon_2}\mathsf{p},$$

and the generators

$$\begin{aligned}
l_0 &= \sum_{m\geqslant 1} d_{-m}d_m + \epsilon_1\epsilon_2\sum_{k\geqslant\frac{1}{2}} k\,\psi_{-k}\psi_k + \frac{1}{8}(\epsilon_1+\epsilon_2)^2 + \frac{1}{2}\mathsf{q}^2 = \epsilon_1\epsilon_2 L_0(p),\\
l_n &= \frac{1}{2}\sum_{m\neq 0,n} d_{n-m}d_m + \frac{\epsilon_1\epsilon_2}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} k\,\psi_{n-k}\psi_k + \frac{1}{2}\left(in\epsilon_1 + in\epsilon_2 + 2\mathsf{q}\right)d_n \qquad \text{(A.13)}\\
&= \epsilon_1\epsilon_2 L_n(p), \qquad \text{(A.14)}\\
g_k &= \sum_{m\neq 0} d_m\psi_{k-m} + (ik\epsilon_1 + ik\epsilon_2 + \mathsf{q})\phi_k = \sqrt{\epsilon_1\epsilon_2}G_k(p).
\end{aligned}$$

They satisfy the modified commutation relations

$$\begin{aligned}
[d_m, d_n] &= m\epsilon_1\epsilon_2\delta_{m+n},\\
[l_m, l_n] &= (m-n)\epsilon_1\epsilon_2 l_{m+n} + \frac{\epsilon_1\epsilon_2(\epsilon_1\epsilon_2 + (\epsilon_1+\epsilon_2)^2)}{12}m\left(m^2-1\right)\delta_{m+n},\\
[l_m, g_k] &= \frac{m-2k}{2}\epsilon_1\epsilon_2 g_{m+k}, \qquad \text{(A.15)}\\
\{g_k, g_l\} &= 2l_{k+l} + \frac{(\epsilon_1\epsilon_2 + (\epsilon_1+\epsilon_2)^2)}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}.
\end{aligned}$$

Since

$$\begin{aligned} l_{-N}g_{-L}\,|p\rangle &= \sum \frac{\langle p|\,\psi^\dagger_{-K}d^\dagger_{-M}l_{-N}g_{-L}\,|p\rangle}{m^n_{NL}} d_{-M}\psi_{-K}\,|p\rangle\,, \\ m^n_{MK} &= \langle p|\,\psi^\dagger_{-K}d^\dagger_{-M}d_{-M}\psi_{-K}\,|p\rangle = (\epsilon_1\epsilon_2)^{\#M}N_{MK}, \end{aligned}$$

one has

$$d_{-M}\psi_{-K}\,|p\rangle = \sum s^n(q)^{-1}_{NL,MK}\, l_{-N}g_{-L}\,|p\rangle \tag{A.16}$$

where

$$s(q)^n_{MK,NL} = \frac{\langle p|\,\psi^\dagger_{-K}d^\dagger_{-M}l_{-N}g_{-L}\,|p\rangle}{(\epsilon_1\epsilon_2)^{\#M}N_{MK}}. \tag{A.17}$$

We shall show that matrix elements $s(q)^n_{MK,NL}$ are polynomials in all variables $\epsilon_1, \epsilon_2, q$. Using the commutation relations

$$\begin{aligned} [l_n, d_m] &= -m(1-\delta_{m+n})\epsilon_1\epsilon_2 d_{m+n} - \frac{1}{2}m\epsilon_1\epsilon_2\big(2q - im\epsilon_1 - im\epsilon_2\big)\delta_{n+m}, \\ [l_n, \psi_s] &= -\left(\frac{1}{2}n + s\right)\epsilon_1\epsilon_2\psi_{n+s}, \\ [g_k, d_m] &= -m\epsilon_1\epsilon_2\psi_{k+m}, \\ \{g_k, \psi_l\} &= \sqrt{\epsilon_1\epsilon_2}(1-\delta_{k+l})d_{k+l} + \sqrt{\epsilon_1\epsilon_2}\big(q - il\epsilon_1 - il\epsilon_2\big)\delta_{r+s}, \end{aligned}$$

one can calculate the numerator of (A.17)

$$\langle p|\,\psi^\dagger_{-K}d^\dagger_{-M}l_{-N}g_{-L}\,|p\rangle$$

moving the bosonic oscillators $d^\dagger_{-M} = d_{m_1}\ldots d_{m_j}$ one by one to the right. Moving d_{m_j} gives a sum of terms, each term containing the commutator of d_{m_j} with l_n or g_l and hence the factor $\epsilon_1\epsilon_2$. Moving subsequent oscillators $d_{m_{j'}}, j' < j$ to the right yields new terms involving commutators with generators, yielding again the factor $\epsilon_1\epsilon_2$, and oscillators resulting from the previous steps. The commutation with the oscillators is nonzero only if $d_{m_{j-1}}$ meets its conjugated counterpart and gives the factor $\epsilon_1\epsilon_2$. Each step contributes therefore the factor $\epsilon_1\epsilon_2$. If after moving all bosonic oscillators to the right the result is nonzero it has the overall factor $\epsilon_1^j\epsilon_2^j$ which cancels against the denominator of (A.17).

Consider now the equation

$$\omega(q,j)\psi_{-K}\,|p\rangle = \sum \omega(q,j)s^n(q)^{-1}_{NL,\emptyset K}\, l_{-N}g_{-L}\,|p\rangle \tag{A.18}$$

where

$$\omega(q,j) = \prod_{\substack{1\leqslant n,m \\ n+m\leqslant 2j \\ n+m\,\in\, 2\mathbb{N}}} \big(2qi + (n-1)\epsilon_1 + (m-1)\epsilon_2\big)\,.$$

Under the scaling

$$d_n \to \lambda d_n, \quad \psi_l \to \psi_l, \quad \epsilon_1 \to \lambda\epsilon_1, \quad \epsilon_2 \to \lambda\epsilon_2,$$

one has the homogeneous transformation laws

$$l_n \to \lambda^2 l_n, \quad g_l \to \lambda g_l, \quad \omega(q,j) \to \lambda^{j^2}\omega(q,j).$$

As the r.h.s. of (A.18) is a combination of linearly independent vectors and the l.h.s. scales as λ^{j^2} the scaling rule for each coefficient must be

$$\omega(q,j)s^n(q)^{-1}_{NL,\emptyset K} \to \lambda^{j^2-2\#N-\#L}\omega(q,j)s^n(q)^{-1}_{NL,\emptyset K}.$$

If $\omega(q,j)s^n(q)^{-1}_{NL,\emptyset K}$ is a polynomial in variables $q, \epsilon_1, \epsilon_2$ then the scaling implies

$$\deg_p\left(\Omega(p,j)S^n(p)^{-1}_{NL,\emptyset K}\right) = \deg_q\left(\omega(q,j)s^n(q)^{-1}_{NL,\emptyset K}\right) \leq j^2 - 2\#N - \#L.$$

We shall show that this is indeed the case. To this end let us first calculate the determinant of s^n. Since,

$$S^n_{NL,MK}(p) = (\epsilon_1\epsilon_2)^{\frac{1}{2}\#N-\#M-\frac{1}{2}\#K}s^n(q)_{NL,MK}$$

it follows from relation (2.13) that the whole q dependence of $\det s^n$ is given by the factor

$$\prod_{\substack{1\leqslant rs\leqslant 2n\\ r+s\in 2\mathbb{N}}}(2q - ir\epsilon_1 - is\epsilon_2)^{P_{\rm NS}(n-\frac{rs}{2})}.$$

In order to find the full ϵ_1, ϵ_2 dependence we calculate the determinant of the matrix

$$b^n_{M'K',MK} = \langle\Delta_p|\, g^\dagger_{-K'} l^\dagger_{-M'} l_{-M} g_{-K}\, |\Delta_p\rangle\,.$$

Up to a numerical factor it is given by

$$\det b^n \propto \det m^n \prod_{\substack{1\leqslant rs\leqslant 2n\\ r+s\in 2\mathbb{N}}}\left((2q - ir\epsilon_1 - is\epsilon_2)\,(2q + ir\epsilon_1 + is\epsilon_2)\right)^{P_{\rm NS}(n-\frac{rs}{2})}. \tag{A.19}$$

where m^n is the diagonal matrix $m^n_{M'K',MK} = m^n_{MK}\delta_{M'K',MK}$. This easily follows from the Kac formula for matrix B^n

$$\det B^n \propto \prod_{\substack{1\leqslant rs\leqslant 2n\\ r+s\in 2\mathbb{N}}}\left(\left(2p - irb - is\frac{1}{b}\right)\left(2p + irb + is\frac{1}{b}\right)\right)^{P_{\rm NS}(n-\frac{rs}{2})}$$

and the relations

$$\begin{aligned} B^n(\Delta_p)_{M'K',MK} &= (\epsilon_1\epsilon_2)^{-\#M'-\frac{1}{2}\#K'-\#M-\frac{1}{2}\#K}b^n_{M'K',MK},\\ \deg_{\Delta_p} B^n &= \sum(\#M + \#K). \end{aligned}$$

Since

$$b^n = (s^n)^\dagger m^n s^n$$

formula (A.19) implies that up to a numerical factor

$$\det(s^n) \propto \prod_{\substack{1\leqslant rs\leqslant 2n\\ r+s\in 2\mathbb{N}}}(2q - ir\epsilon_1 - is\epsilon_2)^{P_{\rm NS}(n-\frac{rs}{2})}.$$

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The matrix elements $s^n(q)^{-1}_{NL,\emptyset K}$ are given by

$$s^n(q)^{-1}_{NL,\emptyset K} = \frac{C[s^n_{\emptyset K,NL}]}{\det s^n} \tag{A.20}$$

where $C[s^n_{\emptyset K,NL}]$ denotes the cofactor of the matrix element $s^n_{\emptyset K,NL}$. $C[s^n_{\emptyset K,NL}]$ is a sum o products of matrix elements $s^n_{M'K',N'L'}$ and therefore a polynomial in variables $\epsilon_1, \epsilon_2, q$. It follows from Prop.1 that the only singularities of the coefficients $S^n(p)^{-1}_{NL,\emptyset K}$ are simple poles at zeros of $\Omega(p,j)$. The same is true for $s^n(q)^{-1}_{NL,\emptyset K}$ and $\omega(q,j)$. But this means cancelation of many factors between the nominator and the denominator in (A.20). The only possible linear factors left in the denominator are those entering $\omega(q,j)$, hence

$$\omega(q,j) s^n(q)^{-1}_{NL,\emptyset K}$$

is a polynomial in all variables.

B Generalized Selberg integral

Our task is to calculate the integral

$$I_N(A,B;g) = \int_0^1 dt_1 \ldots \int_0^1 dt_N \prod_{k=1}^{N} t_k^A (1-t_k)^B \prod_{1\leqslant k<l\leqslant N} |t_k - t_l|^{2g}\, \boldsymbol{P}_N(t_1,\ldots,t_N) \tag{B.1}$$

for even $N = 2m$ and where

$$\boldsymbol{P}_N(t_1,\ldots,t_N) = \langle \psi(t_1)\ldots\psi(t_N)\rangle \prod_{1\leqslant k<l\leqslant N} (t_k - t_l) = \mathrm{pf}\,(\boldsymbol{A}) \prod_{1\leqslant k<l\leqslant N} (t_k - t_l), \tag{B.2}$$

with

$$A^k{}_l = \begin{cases} 0 & \text{for } k = l, \\ \frac{1}{t_k - t_l} & \text{for } k \neq l, \end{cases}$$

is a symmetric polynomial. Our derivation parallels the original Selberg method (see [35] for a pedagogical discussion of various methods of deriving the Selberg formula). Integral (B.1) has been already calculated long time ago (although in a different way) in [36, 37]. It is also a special case of a more general formula discussed in [31]. We present a simpler derivation mainly for the completeness of the present paper.

Since for $t_1 \to t_2$:

$$\langle \psi(t_1)\psi(t_2)\psi(t_3)\ldots\psi(t_N)\rangle = \frac{1}{t_1 - t_2}\langle \psi(t_3)\ldots\psi(t_N)\rangle + \text{finite},$$

the polynomials $\boldsymbol{P}_N$ satisfy the "clustering property":

$$\boldsymbol{P}_N(x,x,t_1,\ldots,t_{N-2}) = \prod_{k=1}^{N-2}(x-t_k)^2 \boldsymbol{P}_{N-2}(t_1,\ldots,t_{N-2}) \tag{B.3}$$

which is an important ingredient of the presented calculation.

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Let g be a natural number. Then,

$$\boldsymbol{D}_N(t_1,\ldots,t_N) = \prod_{1\leqslant k<l\leqslant N} (t_k - t_l)^{2g}\; \boldsymbol{P}_N(t_1,\ldots,t_N) \tag{B.4}$$

is a symmetric, uniform polynomial. From (B.4) and (B.2) one infers its behavior under scaling and inverting of all arguments

$$\boldsymbol{D}_N(\Lambda t_1,\ldots,\Lambda t_N) = \Lambda^{N(N-1)g+\frac{1}{2}N(N-2)}\, \boldsymbol{D}_N(t_1,\ldots,t_N) \tag{B.5}$$

$$\boldsymbol{D}_N(t_1^{-1},\ldots,t_N^{-1}) = \prod_{i=1}^{N} t_i^{1-(N-1)(2g+1)}\, \boldsymbol{D}_N(t_1,\ldots,t_N). \tag{B.6}$$

The polynomial $\boldsymbol{D}_N$ can be presented as

$$\boldsymbol{D}_N(t_1,\ldots,t_N) = \sum_{\nu_j} c_{\nu_1,\ldots,\nu_N}\, t_1^{\nu_1}\ldots t_N^{\nu_N}. \tag{B.7}$$

Since the expansion coefficients $c_{\nu_1,\ldots,\nu_N}$ are totally symmetric with respect to permutation of their indices, we can rewrite the sum as

$$\boldsymbol{D}_N(t_1,\ldots,t_N) = \sum_{\nu_j} c_{\nu_1,\ldots,\nu_N} t_{(1}^{\nu_1}\ldots t_{N)}^{\nu_N}$$

where the indices ν_k are ordered

$$\nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_N \tag{B.8}$$

and the bracket denotes symmetrization.

The crucial point of the Selberg method is to find a lower and an upper bound on the possible values of ν_k. Scaling property (B.5) gives

$$\sum_{k=1}^{N} \nu_k = N(N-1)g + \frac{1}{2}N(N-2) \tag{B.9}$$

and together with (B.8) yields

$$\nu_N \geqslant (N-1)g + \frac{1}{2}(N-2). \tag{B.10}$$

One also has

$$\sum_{l=1}^{N-k} \nu_l + \nu_k^{\max} \geqslant N(N-1)g + \frac{1}{2}N(N-2), \qquad k = 0,1,\ldots,N-1$$

where $\nu_0^{\max} = 0$ and for $k > 0$, $\nu_k^{\max}$ denotes the maximal joint degree of the polynomial $\boldsymbol{D}_N$ in the variables $t_{N-k+1},\ldots t_N$:

$$\boldsymbol{D}_N(t_1,\ldots,t_{N-k},\Lambda t_{N-k+1},\ldots,\Lambda t_N) = \Lambda^{\nu_k^{\max}}\boldsymbol{Q}_N(t_1,\ldots,t_N) + \mathcal{O}\big(\Lambda^{\nu_k^{\max}-1}\big).$$

Let

$$N = 2m, \qquad k = N - 2p - j, \quad p = 0, \ldots, m-1, \quad j = 1, 2 \tag{B.11}$$

then

$$\begin{aligned} \nu_k^{\max} &= (2m-2p-1)(m+p)(2g+1) - m + p && \text{for } j = 1, \\ \nu_k^{\max} &= 2(m-p-1)(2m+2p+1)g + 2(m-p-1)(m+p) && \text{for } j = 2. \end{aligned} \tag{B.12}$$

This gives

$$\begin{aligned} \sum_{l=1}^{2p+1} \nu_l &\geqslant 2m(2m-1)g+2m(m-1)-(2m-2p-1)(m+p)(2g+1)+m-p \\ &= (2p+1)\left((2p+1-1)g+p-\frac{p}{2p+1}\right) \geqslant (2p+1)\Big((2p+1-1)g+p\Big), \\ \sum_{l=1}^{2p+2} \nu_l &\geqslant 2m(2m-1)g+2m(m-1)-2(m-p-1)(2m+2p+1)g-2(m-p-1)(m+p) \\ &= (2p+2)\Big((2p+2-1)g+p\Big), \end{aligned}$$

or, equivalently

$$\sum_{l=1}^{2p+j} \nu_l \geqslant (2p+j)\Big((2p+j-1)g+p\Big). \tag{B.13}$$

Taking into account the ordering of the indices ν_l we thus get

$$\nu_{2p+j} \geqslant (2p+j-1)g+p \tag{B.14}$$

which for $p = m-1,\ j = 2$ agrees with (B.10).

To obtain the upper bound on the index ν_{2p+j} we use property (B.6). It gives

$$\begin{aligned} \boldsymbol{D}_N(t_1, \ldots, t_N) &= \prod_{j=1}^{N} t_j^{(N-1)(2g+1)-1} \, \boldsymbol{D}_N(t_1^{-1}, \ldots, t_N^{-1}) \\ &= \prod_{j=1}^{N} t_j^{(N-1)(2g+1)-1} \sum_{\nu_1 \leqslant \ldots \leqslant \nu_N} c_{\nu_1,\ldots,\nu_N} t_{(1}^{-\nu_1} \ldots t_{N)}^{-\nu_N} \\ &= \sum_{\nu_1 \leqslant \ldots \leqslant \nu_N} c_{\nu_1,\ldots,\nu_N} t_{(1}^{\nu'_N} \ldots t_{N)}^{\nu'_1} = \sum_{\nu'_1 \leqslant \ldots \leqslant \nu'_N} c_{\nu'_1,\ldots,\nu'_N} t_{(1}^{\nu'_1} \ldots t_{N)}^{\nu'_N}, \end{aligned}$$

where

$$\nu'_l = (N-1)(2g+1) - 1 - \nu_{N+1-l}.$$

Inequality (B.14) thus yields

$$\nu_{2p+j} \leqslant (m+p-1) + \big(2(m+p)+j-2\big)g. \tag{B.15}$$

Upon inserting (B.7) into (B.1), the well-known formula for the Euler Beta gives

$$I_N(A,B;g) = \sum_{\{\nu_k\}} c_{\nu_1,\dots\nu_N} \prod_{k=1}^{N} \int_0^1 t_k^{\nu_i+A}(1-t_k)^B\, dt_k = \sum_{\{\nu_k\}} c_{\nu_1,\dots\nu_N} \prod_{k=1}^{N} \frac{\Gamma(1+A+\nu_k)\Gamma(1+B)}{\Gamma(2+A+B+\nu_k)}$$

$$= \sum_{\nu_1\leqslant\dots\leqslant\nu_N} c_{\nu_{(1,\dots\nu_N)}} \prod_{k=1}^{N} \frac{\Gamma(1+A+\nu_k)\Gamma(1+B)}{\Gamma(2+A+B+\nu_k)}.$$

Using (B.14) we get

$$\Gamma(1+A+\nu_{2p+j}) = \frac{\Gamma(1+A+\nu_{2p+j})}{\Gamma(1+A+(2p+j-1)g+p)}\Gamma(1+A+(2p+j-1)g+p)$$

where

$$\frac{\Gamma(1+A+\nu_{2p+j})}{\Gamma(1+A+(2p+j-1)g+p)} = \big(A+\nu_{2p+j}\big)\big(A+\nu_{2p+j}-1\big)\dots\big(1+A+(2p+j-1)g+p\big)$$

is a polynomial in A of degree $\nu_{2p+j}-(2p+j-1)g-p$. This gives

$$\prod_{p=0}^{m-1}\prod_{j=1}^{2}\Gamma(1+A+\nu_{2p+j}) = w_{\nu_1,\dots,\nu_N}(A)\prod_{p=0}^{m-1}\prod_{j=1}^{2}\Gamma(1+A+(2p+j-1)g+p) \quad \text{(B.16)}$$

where

$$w_{\nu_1,\dots,\nu_N}(A) = \prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+A+\nu_{2p+j})}{\Gamma(1+A+(2p+j-1)g+p)}$$

is a polynomial in A of degree (see (B.9)):

$$\sum_{p=0}^{m-1}\sum_{j=1}^{2}\big(\nu_{2p+j}-(2p+j-1)g-p\big) = m(2m-1)g+m(m-1).$$

Similarly, applying (B.15) we get

$$\prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{1}{\Gamma(2+A+B+\nu_{2p+j})} = u_{\nu_1,\dots\nu_N}(A+B) \quad \text{(B.17)}$$

$$\times\prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{1}{\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)}$$

where

$$u_{\nu_1,\dots\nu_N}(A+B) = \prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)}{\Gamma(2+A+B+\nu_{2p+j})}$$

is a polynomial in A,B, of degree

$$\sum_{p=0}^{m-1}\sum_{j=1}^{2}\big((m+p-1)+(2(m+p)+j-2)g-\nu_{2p+j}\big) = m(2m-1)g+m(m-1).$$

Finally, we can write

$$\prod_{p=0}^{m-1}\prod_{j=1}^{2}\Gamma(1+B) = \frac{1}{Q_m(B)}\prod_{p=0}^{m-1}\prod_{j=1}^{2}\Gamma(1+B+(2p+j-1)g+p) \tag{B.18}$$

where

$$Q_m(B) = \prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+B+(2p+j-1)g+p)}{\Gamma(1+B)}$$

is a polynomial in B of degree

$$\sum_{p=0}^{m-1}\sum_{j=1}^{2}\big((2p+j-1)g+p\big) = m(2m-1)g+m(m-1).$$

Using (B.16)–(B.18) we thus get

$$\prod_{k=1}^{N}\frac{\Gamma(1+A+\nu_k)\Gamma(1+B)}{\Gamma(2+A+B+\nu_k)} = \frac{w_{\nu_1,\dots,\nu_N}(A)u_{\nu_1,\dots,\nu_N}(A+B)}{Q_m(B)}$$
$$\times\prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+A+(2p+j-1)g+p)\Gamma(1+B+(2p+j-1)g+p)}{\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)}$$

and consequently

$$I_N(A,B;g)=\frac{P_m(A,B)}{Q_m(B)}\prod_{p=0}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+A+(2p+j-1)g+p)\Gamma(1+B+(2p+j-1)g+p)}{\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)} \tag{B.19}$$

where

$$P_m(A,B) = \sum_{\nu_1\leqslant\dots\leqslant\nu_N} c_{\nu_{(1,\dots,\nu_N)}}\, w_{\nu_1,\dots,\nu_N}(A)u_{\nu_1,\dots,\nu_N}(A+B).$$

Since the coefficients $c_{\nu_1,\dots,\nu_N}$ do not depend on A and B, $P_m(A,B)$ is a polynomial in A of degree not greater than $2m(2m-1)g+2m(m-1)$ and a polynomial in B of degree not greater than $m(2m-1)g+m(m-1)$.

In view of an obvious symmetry of integral (B.1) with respect to the exchange of A and B,

$$I_N(A,B;g) = I_N(B,A;g),$$

formula (B.19) gives

$$\frac{P_m(A,B)}{Q_m(B)} = \frac{P_m(B,A)}{Q_m(A)} \tag{B.20}$$

The r.h.s. of (B.20) is a polynomial in B, the same must be therefore true for the l.h.s. Since the degree of $P_m(A,B)$ as a polynomial in B is bounded by the degree of $Q_m(B)$, this polynomial is of a zero degree and thus $B-$independent. In the same way one shows that ratio (B.20) is also $A-$independent. We can thus write

$$\frac{P_m(A,B)}{Q_m(B)} = C_m(g)$$

and consequently

$$I_N(A,B;g) = C_m(g) \prod_{p=0}^{m-1}\prod_{j=1}^{2} \frac{\Gamma(1+A+(2p+j-1)g+p)\Gamma(1+B+(2p+j-1)g+p)}{\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)}. \tag{B.21}$$

Our next task it to derive (and solve) a recurrence relation for $C_m(g)$. To this end let us write

$$I_N(A,B;g) = N! \int_0^1 dt_1 \int_{t_1}^1 dt_2 \ldots \int_{t_{N-1}}^1 dt_N\, \boldsymbol{P}_N(t_1,\ldots,t_N) \prod_{k=1}^{N} t_k^A(1-t_k)^B \prod_{1\leqslant k<l\leqslant N}(t_l-t_k)^{2g} \tag{B.22}$$

and change the integration variables in (B.22):

$$t_1 = \tau\xi_1,\quad t_2 = \tau\xi_2,\quad \xi_1+\xi_2 = 1,\quad t_i \to t_{i-2},\quad i = 3,4,\ldots N.$$

Since

$$\int_0^1 dt_1 \int_{t_1}^1 dt_2\, f(t_1,t_2) = \int_0^1 d\tau\,\tau \int_0^{\frac{1}{2}} d\xi_1\, f(\tau\xi_1,\tau\xi_2) + \int_1^2 d\tau\,\tau \int_{1-\frac{1}{\tau}}^{\frac{1}{2}} d\xi_1\, f(\tau\xi_1,\tau\xi_2), \quad \xi_2 = 1-\xi_1,$$

we get

$$\begin{aligned} I_N(A,B;g) = N! &\int_0^1 d\tau\,\tau^{2A+2g+1} \int_0^{\frac{1}{2}} d\xi_1 \prod_{j=1}^{2} \xi_j^A(1-\tau\xi_j)^B|\xi_1-\xi_2|^{2g}\, J_{N-2}(A,B;g|\tau,\xi_j) \\ &+ N! \int_1^2 d\tau\,\tau^{2A+2g+1} \int_{1-\frac{1}{\tau}}^{\frac{1}{2}} d\xi_1 \prod_{j=1}^{2} \xi_j^A(1-\tau\xi_j)^B|\xi_1-\xi_2|^{2g}\, J_{N-2}(A,B;g|\tau,\xi_j) \end{aligned}$$

where

$$\begin{aligned} J_{N-2}(A,B;g|\tau,\xi_j) = \int_{\tau\xi_2}^1 dt_1 \int_{t_1}^1 dt_2 \ldots \int_{t_{N-3}}^1 dt_{N-2} \prod_{k=1}^{N-2} t_k^A(1-t_k)^B \prod_{j=1}^{2}|t_k-\tau\xi_j|^{2g} \prod_{1=k<l}^{N-2}|t_k-t_l|^{2g} \\ \times\ \boldsymbol{P}_N(\tau\xi_1,\tau\xi_2,t_1,\ldots,t_{N-2}). \end{aligned}$$

Applying the identity

$$\lim_{a\to(-1)^+}(1+a)\int_0^\Lambda dx\, x^a f(x) = f(0) \tag{B.23}$$

we get

$$\lim_{A\to -g-1}(A+g+1)I_N(A,B;g) \,=\, N!\,X_2(g)\,J_{N-2}(-g-1,B;g|0,\xi_j) \tag{B.24}$$

where

$$X_2(g) \,=\, \frac{1}{2}\int_0^{\frac{1}{2}} d\xi_1 \prod_{j=1}^{2} \xi_j^{-g-1}\,|\xi_1-\xi_2|^{2g} = \frac{1}{4}\int_0^1 d\xi_1 \prod_{j=1}^{2} \xi_j^{-g-1}\,|\xi_1-\xi_2|^{2g}, \qquad \xi_2 = 1-\xi_1,$$

and we applied the clustering property of the polynomial $\boldsymbol{P}_N$. Further

$$\begin{aligned} J_{N-2}(-g-1,B;g|0,\xi_j) \,&=\, \int_0^1 dt_1 \ldots \int_{t_{N-3}}^1 dt_{N-2} \prod_{k=1}^{N-2} t_k^{1+3g}(1-t_k)^B \prod_{1=k<l}^{N-2} |t_k-t_l|^{2g} \boldsymbol{P}_{N-2}(t_1,\ldots,t_{N-2}) \\ &=\, \frac{1}{(N-2)!} I_{N-2}(1+3g,B;g). \end{aligned}$$

To determine the value of $X_2(g)$ note that, since

$$\boldsymbol{P}_2(t_1,t_2) = 1,$$

the integral $I_2(A,B;g)$ is just a "standard" Selberg integral with a well-known value

$$I_2(A,B;g) \,=\, S_2(A,B;g) = \prod_{j=0}^{1} \frac{\Gamma(A+1+jg)\Gamma(B+1+jg)\Gamma(1+(j+1)g)}{\Gamma\big(A+B+2+(1+j)g\big)\Gamma(1+g)},$$

and consequently

$$X_2(g) \,=\, \frac{1}{2!}\lim_{A\to -g-1}(A+g+1)S_2(A,B;g) = \frac{\Gamma(-g)}{2}\prod_{j=1}^{2}\frac{\Gamma(1+jg)}{\Gamma(1+g)} \,=\, \frac{1}{2}\frac{\Gamma(-g)\Gamma(1+2g)}{\Gamma(1+g)},$$

so that (B.24) takes the form

$$\lim_{A\to -g-1}(A+g+1)I_N(A,B;g) = \frac{N(N-1)}{2}\frac{\Gamma(-g)\Gamma(1+2g)}{\Gamma(1+g)}\, I_{N-2}(1+3g,B;g). \tag{B.25}$$

Using in this equality the r.h.s. of formula (B.21) one gets after a short calculation

$$C_m(g) \,=\, \frac{2m(2m-1)}{2}\frac{\Gamma((2m-1)g+m)\Gamma(2mg+m)}{\Gamma^2(1+g)}\, C_{m-1}(g)$$

which implies

$$C_m(g) \,=\, \frac{(2m)!}{2^m}\prod_{p=1}^{m-1}\prod_{j=1}^{2}\frac{\Gamma(1+p+(2p+j)g)}{\Gamma(1+g)}\, C_1(g).$$

The fact that for $m=1$ integral (B.1) is just a Selberg integral yields

$$C_1(g) = \frac{\Gamma(1+g)\Gamma(1+2g)}{\Gamma^2(1+g)}$$

and

$$C_m(g) = \frac{(2m)!}{2^m} \prod_{p=0}^{m-1} \prod_{j=1}^{2} \frac{\Gamma(1+p+(2p+j)g)}{\Gamma(1+g)}. \tag{B.26}$$

This finally gives

$$\begin{aligned} &\int_0^1 dt_1 \ldots \int_0^1 dt_{2m} \prod_{k=1}^{2m} t_k^A (1-t_k)^B \prod_{1\leqslant k<l\leqslant 2m} |t_k - t_l|^{2g} \, \boldsymbol{P}_{2m}(t_1,\ldots,t_{2m}) \\ &= \frac{(2m)!}{2^m} \prod_{p=0}^{m-1} \prod_{j=1}^{2} \frac{\Gamma(+p+(2p+j)g)\Gamma(1+A+(2p+j-1)g+p)\Gamma(1+B+(2p+j-1)g+p)}{\Gamma(1+g)\Gamma\big(2+A+B+(m+p-1)+(2(m+p)+j-2)g\big)}. \end{aligned} \tag{B.27}$$

Formula (B.27) has been derived for $g \in \mathbb{N}$. One can however resort to the Carlson theorem [38] to check its validity also for complex g, for which the integral is convergent.

Similar methods can be used to calculate the "odd" integral

$$I^{(1)}_{2m-1}(A,B;g) = \int_0^1 dt_1 \ldots \int_0^1 dt_{2m-1} \prod_{i=1}^{2m-1} t_i^A (1-t_i)^B \prod_{1\leqslant k<l\leqslant 2m-1} |t_k - t_l|^{2g} \, \boldsymbol{P}^{(1)}_{2m-1}(t_1,\ldots,t_{2m-1}) \tag{B.28}$$

where

$$\boldsymbol{P}^{(1)}_{2m-1}(t_1,\ldots,t_{2m-1}) = \prod_{1\leqslant k<l\leqslant 2m-1} (t_k - t_l) \, \langle 0_f | \psi(1)\psi(t_1)\ldots\psi(t_{2m-1}) | 0_f \rangle.$$

Since we do not need this integral in the general form, we skip the calculation and just quote the result:

$$\begin{aligned} I^{(1)}_{2m-1}(A,B;g) &= \frac{(2m-1)!}{2^{m-1}} \\ &\times \frac{1}{B} \prod_{p,j} \frac{\Gamma(1+p+(2p+j)g)\Gamma(1+A+(2p+j-1)g+p)\Gamma(1+B+(2p+j-1)g+p)}{\Gamma(1+g)\Gamma\,\big(2+A+B+(m+p+j-3)+\big(2(m+p)+j-3\big)g\big)} \end{aligned} \tag{B.29}$$

where for $p = 0, 1, \ldots, m-2$ we have $j = 1, 2$ while for $p = m-1$ we have $j = 1$.

C Gamma Barnes identities

For $\Re x > 0$ the Barnes double gamma function $\Gamma_b(x)$ can be defined by the integral representation [39, 19]

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Q}{2}t}}{\left(1-e^{-tb}\right)\left(1-e^{-t/b}\right)} - \frac{\left(\frac{Q}{2}-x\right)^2}{2e^t} - \frac{\frac{Q}{2}-x}{t} \right].$$

It can be analytically continued to the whole complex x plane as a meromorphic function with poles located at $x = -mb - n\frac{1}{b}$, $m, n \in \mathbb{N}$.

From the shift formulae

$$\Gamma_b(x+b) = \sqrt{2\pi}\, b^{bx-\frac{1}{2}}\, \Gamma^{-1}(bx)\, \Gamma_b(x),$$
$$\Gamma_b\left(x+b^{-1}\right) = \sqrt{2\pi}\, b^{-x/b+\frac{1}{2}}\, \Gamma^{-1}(x/b)\, \Gamma_b(x),$$

one easily derives the multiple shift relations

$$\Gamma_b(x+jb) = \frac{(2\pi)^{\frac{j}{2}}\, b^{jbx+\frac{1}{2}j(j-1)b^2-\frac{1}{2}j}}{\left(\prod\limits_{k=0}^{j-1}\Gamma\big((x+kb)b\big)\right)}\Gamma_b(x),$$

$$\Gamma_b(x-jb) = (2\pi)^{-\frac{j}{2}}\, b^{-jbx+\frac{1}{2}j(j+1)b^2+\frac{1}{2}j}\left(\prod_{k=1}^{j}\Gamma\big((x-kb)b\big)\right)\Gamma_b(x),$$

$$\Gamma_b\left(x+jb^{-1}\right) = \frac{(2\pi)^{\frac{j}{2}}\, b^{-\frac{jx}{b}-\frac{1}{2}\frac{j(j-1)}{b^2}+\frac{1}{2}j}}{\left(\prod\limits_{k=0}^{j-1}\Gamma(\frac{x+\frac{k}{b}}{b})\right)}\Gamma_b(x),$$

$$\Gamma_b\left(x-jb^{-1}\right) = (2\pi)^{-\frac{j}{2}}\, b^{\frac{jx}{b}-\frac{1}{2}\frac{j(j+1)}{b^2}-\frac{1}{2}j}\left(\prod_{k=1}^{j}\Gamma(\frac{x-\frac{k}{b}}{b})\right)\Gamma_b(x).$$

In order to verify identity (4.2) one first checks the behavior of its both sides under the shifts $\alpha \to \alpha + 2b$ and $\alpha \to \alpha + 2b^{-1}$. For non-rational b it yields the proof of (4.2) up to α-independent factor:

$$\frac{\Gamma_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(\alpha^{\Gamma}+b^{\Gamma})} = C(b) b^{-\frac{b^2}{1-b^2}\frac{\alpha(Q-\alpha)}{4}}\left(\frac{1-b^2}{2}\right)^{-\frac{\alpha(Q-\alpha)}{8}}\Gamma_b^{\mathrm{NS}}(\alpha)$$

To find $C(b)$ one can calculate both sides at $\alpha = Q$, hence

$$C(b) = \frac{\Gamma_{b^{\mathrm{L}}}(Q^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}((b^{\Gamma})^{-1})\Gamma_b\left(\frac{Q}{2}\right)\Gamma_b(Q)} = \frac{\sqrt{2\pi b^{\Gamma}}\,\Gamma_{b^{\mathrm{L}}}(Q^{\mathrm{L}})}{\Gamma_{b^{\Gamma}}(b^{\Gamma}+(b^{\Gamma})^{-1})\Gamma_b\left(\frac{Q}{2}\right)\Gamma_b(Q)}.$$

Identities (4.3) and (4.4) are obtained from (4.2) by substituting $\alpha \to \alpha + b$ and $\alpha \to \alpha - b$, respectively.

Multiplying (4.2) for α and for $Q - \alpha$ side by side yields the identity for upsilon functions proposed in [15]:

$$\frac{\Upsilon_{b^{\mathrm{L}}}(\alpha^{\mathrm{L}})}{\Upsilon_{b^{\Gamma}}(\alpha^{\Gamma}+b^{\Gamma})} = \frac{\Upsilon_{b^{\Gamma}}(b^{\Gamma})\Upsilon_b(b)\,\Upsilon_b\left(\frac{Q}{2}\right)}{\Upsilon_{b^{\mathrm{L}}}(b^{\mathrm{L}})} b^{-\frac{b^2}{1-b^2}\frac{\alpha(Q-\alpha)}{2}}\left(\frac{1-b^2}{2}\right)^{-\frac{\alpha(Q-\alpha)}{4}+\frac{1}{2}}\Upsilon_b^{\mathrm{NS}}(\alpha).$$

The relations supporting the calculations of ratios of the LL chiral structure constants presented in subsection 4.1 read

$$\begin{aligned}
&\frac{\Gamma_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}+\frac{1}{2}jb^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(\alpha^{\Gamma}+b^{\Gamma}\right)}{\Gamma_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(\alpha^{\Gamma}+b^{\Gamma}+\frac{1}{2}j(b^{\Gamma})^{-1}\right)} \\
&\qquad = \frac{\Gamma_{b^{\mathrm{L}}}\left(Q^{\mathrm{L}}-\alpha^{\mathrm{L}}-\frac{1}{2}jb^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(-\alpha^{\Gamma}+(b^{\Gamma})^{-1}\right)}{\Gamma_{b^{\mathrm{L}}}\left(Q^{\mathrm{L}}-\alpha^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(-\alpha^{\Gamma}-\frac{1}{2}j(b^{\Gamma})^{-1}+(b^{\Gamma})^{-1}\right)} \\
&\qquad = b^{\frac{j(2\alpha b+j-2)}{4(1-b^2)}+\frac{j}{4}}\left(2-2b^2\right)^{-\frac{j^2}{8}}\, l^{\mathrm{NS}}(\alpha,j)
\end{aligned} \tag{C.1}$$

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for $j \in 2\mathbb{N}$ and

$$\frac{\Gamma_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}+\frac{1}{2}jb^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(\alpha^{\Gamma}+\frac{1}{2}(b^{\Gamma})^{-1}+b^{\Gamma}\right)}{\Gamma_{b^{\mathrm{L}}}\left(\alpha^{\mathrm{L}}+\frac{1}{2}b^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(\alpha^{\Gamma}+b^{\Gamma}+\frac{1}{2}j(b^{\Gamma})^{-1}\right)} \tag{C.2}$$
$$= \frac{\Gamma_{b^{\mathrm{L}}}\left(-\alpha^{\mathrm{L}}+b^{\mathrm{L}}+(b^{\mathrm{L}})^{-1}-\frac{1}{2}jb^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(-\alpha^{\Gamma}+\frac{1}{2}(b^{\Gamma})^{-1}\right)}{\Gamma_{b^{\mathrm{L}}}\left(-\alpha^{\mathrm{L}}+(b^{\mathrm{L}})^{-1}+\frac{1}{2}b^{\mathrm{L}}\right)\Gamma_{b^{\Gamma}}\left(-\alpha^{\Gamma}+(b^{\Gamma})^{-1}-\frac{1}{2}j(b^{\Gamma})^{-1}\right)}$$
$$= b^{\frac{(j-1)(2\alpha b+j-1)}{4(1-b^2)}+\frac{j-1}{4}}\left(2-2b^2\right)^{-\frac{j^2-1}{8}}\, l^{\mathrm{R}}(\alpha,j)$$

for $j \in 2\mathbb{N}+1$. The functions $l^{\mathrm{NS}}(\alpha,j)$ and $l^{\mathrm{R}}(\alpha,j)$ are defined by (2.24) and (3.19), respectively. The formulae above can be easily derived using the multiple shift formulae and some simple consequences of definitions (1.2), (3.8)

$$b^{\mathrm{L}}b^{\Gamma}=b,\quad (b^{\Gamma})^{-2}=2+(b^{\mathrm{L}})^2,\quad b^{\mathrm{L}}\alpha^{\mathrm{L}}=(b^{\Gamma})^{-1}\alpha^{\Gamma}=\frac{b\,\alpha}{1-b^2}.$$

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